LM-AUSMPW+ scheme for all-speed on unstructured grid

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Introduction

◆ Characteristics which flow solver needs to have

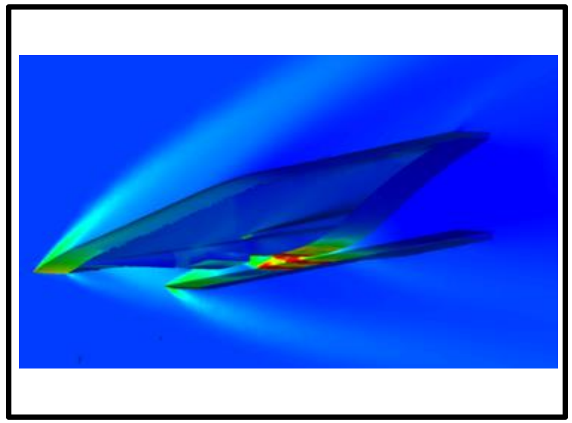
Robustness & Accuracy

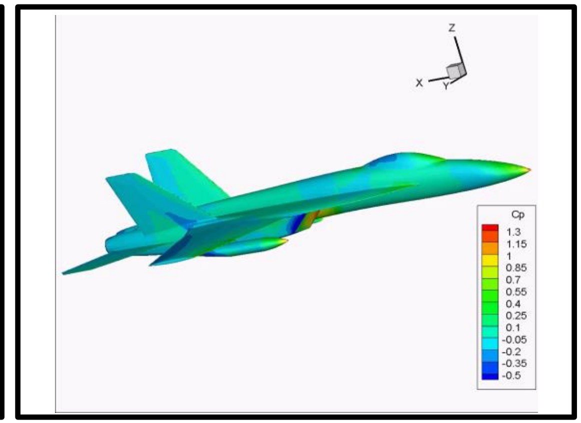
Multi-physics

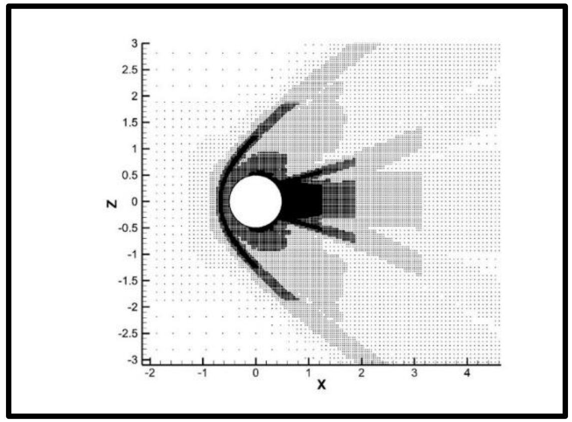
Turbulent flow
Plasma
Ablation
All-speed flow
Moving object analysis

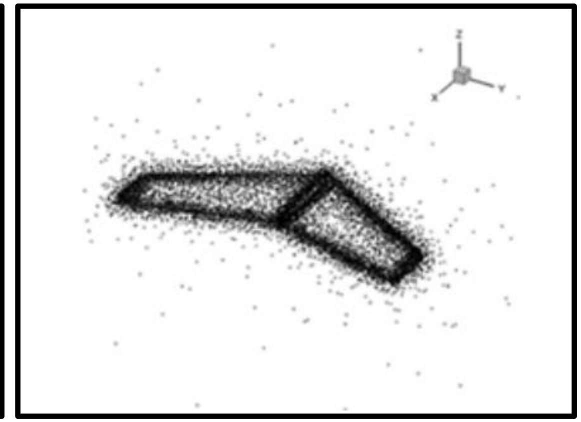
Efficiency

Mesh optimization
Fast convergence



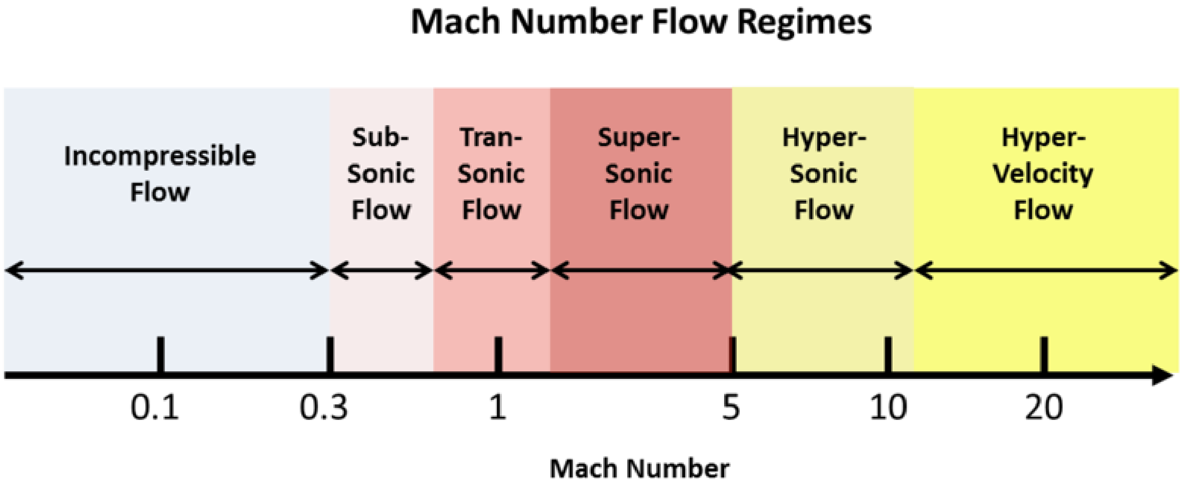


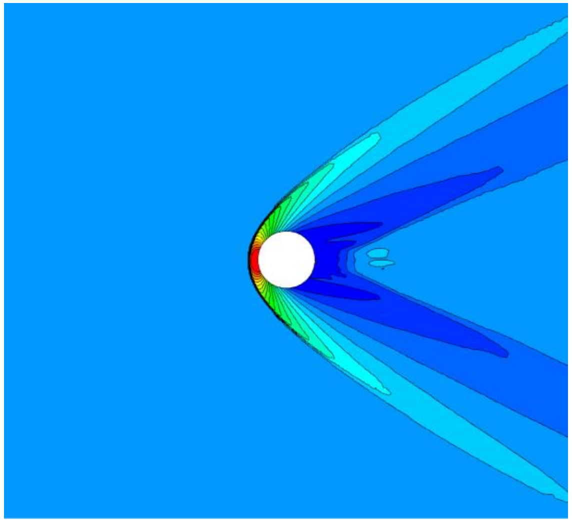




Introduction

- **All-speed flow**
  - **The mathematical properties of the equations vary depending on the flow velocity range.**
  - **It is common for users to use single method for the whole flow field.**
  - **Most flux schemes which successfully solve the accuracy problem at supersonic speed have difficulties in obtaining low Mach number flow solution.**





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Previous study

- Governing equations
- Baseline scheme
- LM-AUSMPW+

Governing equations

Euler equations

- **Conservative form is used for compressible flow**

$$\frac{\partial q}{\partial t} + \nabla \cdot f_c(q) = 0$$

$$q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e_t \end{bmatrix}, \qquad f_c(q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u H \end{bmatrix} \hat{\imath} + \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v H \end{bmatrix} \hat{\jmath}$$

- **Nondimensionalization**

$$\rho^* = \frac{\rho}{\rho_\infty}, u^* = \frac{u}{u_\infty}, v^* = \frac{v}{v_\infty}, x^* = \frac{x}{l_\infty},$$

$$y^* = \frac{y}{l_\infty}, p^* = \frac{p}{\rho_\infty c_\infty^2}, e^* = \frac{e}{c_\infty^2}, t^* = \frac{t}{u_\infty / l_\infty}$$

$$\frac{\partial \rho^*}{\partial t^*} + \nabla(\rho^* u^*) = 0$$

$$\frac{\partial}{\partial t^*} \rho^* u^* + \nabla(\rho^* u^* u^*) = -\frac{1}{M_\infty^2} \nabla p^*$$

$$\frac{\partial}{\partial t^*} \rho^* e^* + \nabla(\rho^* e^* u^* + p^* u^*) = 0$$

Baseline scheme: AUSMPW+/M-AUSMPW+

AUSMPW+

- **Control convection property by considering both left and right states across strong shock ▶ numerical oscillation/overshoot is cured**

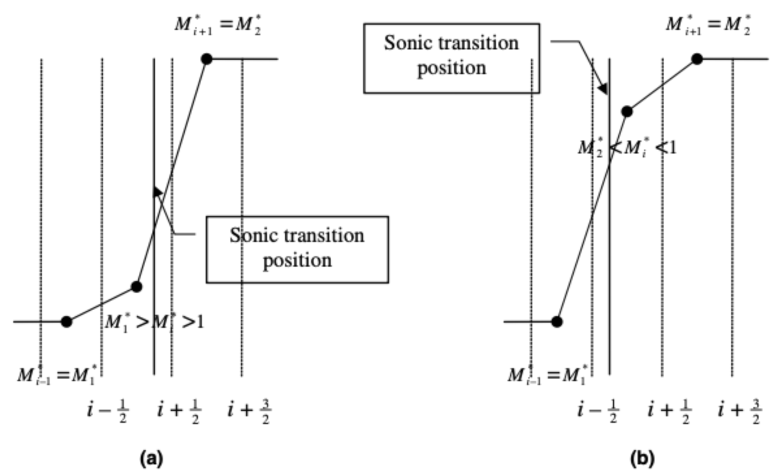
$$F_{\frac{1}{2}} = \bar{M}_L^{+} c_{\frac{1}{2}} \Phi_L + \bar{M}_R^{-} c_{\frac{1}{2}} \Phi_R + P_L^{+} p_L + P_R^{-} p_R$$

$$if\ M_L^{+} + M_R^{-} \geq 0,$$
$$\bar{M}_L^{+} = M_L^{+} + M_R^{-} \cdot [(1-w)(1+f_R) - f_L]$$
$$\bar{M}_R^{-} = M_R^{-} \cdot w(1+f_R)$$

$$if\ M_L^{+} + M_R^{-} < 0,$$
$$\bar{M}_L^{+} = M_L^{+} \cdot w(1+f_L)$$
$$\bar{M}_R^{-} = M_R^{-} + M_L^{+} \cdot [(1-w)(1+f_L) - f_R]$$

$$f_{L,R} = \left(\frac{p_{L,R}}{p_s} - 1\right) \min(1, \frac{\min(p_{1,L}, p_{1,R}, p_{2,L}, p_{2,R})}{\min(p_L, p_R)})$$

$$w = 1 - \min\left(\frac{p_R}{p_L}, \frac{p_L}{p_R}\right)^3$$



Baseline scheme: AUSMPW+/M-AUSMPW+

M-AUSMPW+

$$F_{\frac{1}{2}} = \bar{M}_L^+ c_{\frac{1}{2}} \Phi_{L,\frac{1}{2}} + \bar{M}_R^- c_{\frac{1}{2}} \Phi_{R,\frac{1}{2}} + P_L^+ p_{L,\frac{1}{2}} + P_R^- p_{R,\frac{1}{2}}$$

- **R1) The region of continuity can be distinguished from discontinuity**
- **R2) Monotonic condition should be satisfied**
- **R3) Convective quantity should maintain upwind characteristic in supersonic flow**

$$\Phi_{L,\frac{1}{2}} = \Phi_L + \frac{max(0,(\Phi_R - \Phi_L)(\Phi_{L,sup} - \Phi_L))}{(\Phi_R - \Phi_L)\left|\Phi_{L,sup} - \Phi_L\right|} min\left[a\frac{|\Phi_R - \Phi_L|}{2}, \left|\Phi_{L,sup} - \Phi_L\right|\right]$$

$$\Phi_{R,\frac{1}{2}} = \Phi_R + \frac{max(0,(\Phi_L - \Phi_R)(\Phi_{R,sup} - \Phi_R))}{(\Phi_L - \Phi_R)\left|\Phi_{R,sup} - \Phi_R\right|} min\left[a\frac{|\Phi_L - \Phi_R|}{2}, \left|\Phi_{R,sup} - \Phi_R\right|\right]$$

$$a = 1 - min\left(1, max(|M_L|, |M_R|)\right)^2$$

- **Modification of pressure splitting function which is aimed to improve accuracy in steady shock discontinuity**

$$If\ M_i^* > 1, M_{i+1}^* < 1\ and\ 0 < M_i^* M_{i+1}^* < 1$$

$$P_{i+1}^- = \max(0, \min\left(0.5, 1 - \frac{\rho_i U_i (U_i - U_{i+1}) + p_i}{p_{i+1}}\right))$$

$$If\ M_i^* > -1, M_{i+1}^* < -1\ and\ 0 < M_i^* M_{i+1}^* < 1$$

$$P_i^+ = \max(0, \min\left(0.5, 1 - \frac{\rho_{i+1} U_{i+1} (U_{i+1} - U_i) + p_{i+1}}{p_i}\right))$$

LM-AUSMPW+ scheme

- **New version of M-AUSMPW+ scheme for low-speed accuracy**
  - **Manipulation of artificial dissipation based on the asymptotic analysis**

Governing equations

$$p^*(x,t) = p_0^*(t) + M_\infty^2 p_2^*(x,t) + \cdots \quad - \ satisfied?$$

AUSMPW+

1) $Order\ of\ 1/M_\infty^2$

$$p^*_{0,i+1,j} - p^*_{0,i-1,j} = 0$$
$$p^*_{0,i,j+1} - p^*_{0,i,j-1} = 0$$

2) $Order\ of\ 1/M_\infty$

$$\frac{1}{2}\sum_l p_{1,l}(\overrightarrow{n_{il}})_x - \frac{3}{4}\sum_l \left[\frac{u_{0,l}\cdot\overrightarrow{n_{x,il}}}{c_{ij}}p_{0,l}\right] = 0$$
$$\frac{1}{2}\sum_l p_{1,l}(\overrightarrow{n_{il}})_y - \frac{3}{4}\sum_l \left[\frac{u_{0,l}\cdot\overrightarrow{n_{y,il}}}{c_{ij}}p_{0,l}\right] = 0$$

M-AUSMPW+

1) $Order\ of\ 1/M_\infty^2$

$$p^*_{0,i+1,j} - p^*_{0,i-1,j} = 0$$
$$p^*_{0,i,j+1} - p^*_{0,i,j-1} = 0$$

2) $Order\ of\ 1/M_\infty$

$$p^*_{1,i+1,j} - p^*_{1,i-1,j} = 0$$
$$p^*_{1,i,j+1} - p^*_{1,i,j-1} = 0$$

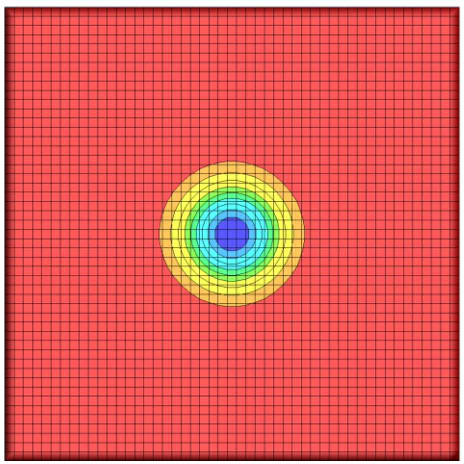
LM-AUSMPW+ scheme

- **New version of M-AUSMPW+ scheme for low-speed accuracy**
  - **Convective term of LM-AUSMPW should satisfy following requirements**
    - C1) Asymptotic characteristics of governing equations should be satisfied in additional numerical dissipation.
    - C2) Supersonic characteristics of baseline scheme should be maintained.
    - C3) The size of numerical dissipation is determined according to Mach number region.
  - **The order of numerical dissipation is manipulated to satisfy asymptotic behavior of governing equations at low mach number region**
  - **Additional numerical dissipation is added to pressure term tied with momentum equations to cure the instability due to zero numerical dissipation**

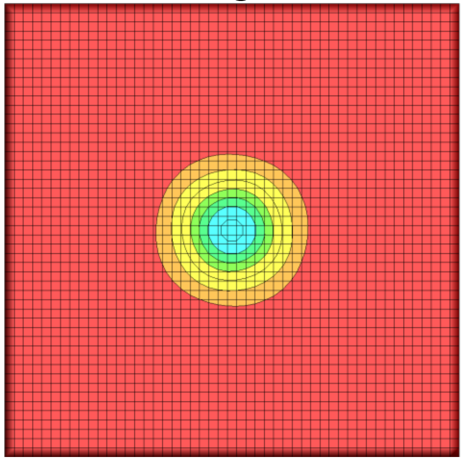
$$p_{\frac{1}{2}} = \frac{p_L + p_R}{2} + \left(P_L^+ p_L + P_R^- p_R - \frac{p_L + p_R}{2}\right)\min(1, M)$$
$$+P_L^+ P_R^- M^2 \left(1 - min\left(1, max(|M_L|, |M_R|)\right)^2\right)(p_R - p_L)$$

LM-AUSMPW+ scheme

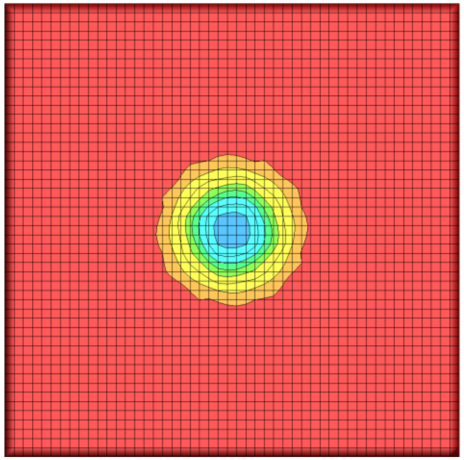
- **New version of M-AUSMPW+ scheme for low-speed accuracy**
  - $Low\ numerical\ dissipation\ in\ the\ region\ where\ Mach\ number\ is\ under\ 1$



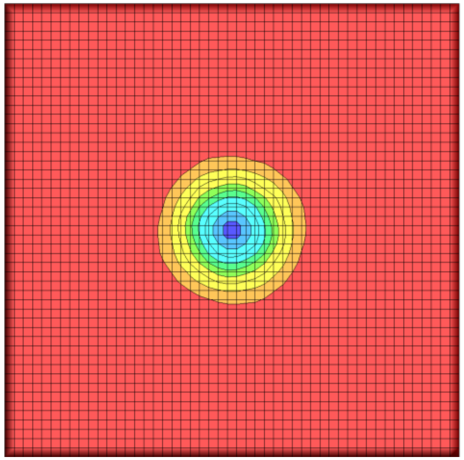
$Initial$



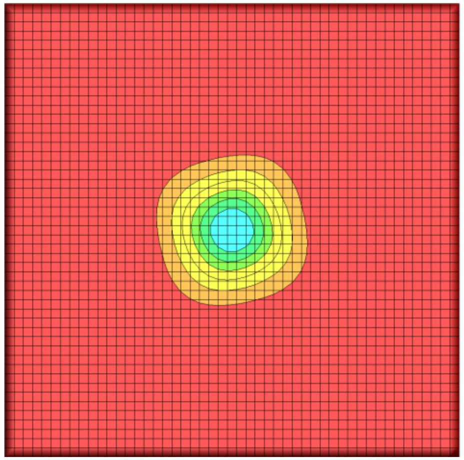
$AUSMPW+$



$M-AUSMPW+$



$LM-AUSMPW+$



$HLLC$

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3. Improvement on scheme

Limitation of LM-AUSMPW+ scheme

- **LM-AUSMPW+ of previous study has 2 limitations**
  - **Accuracy problem**
    - **Artificial viscosity is always decreased in LM-AUSMPW+ scheme, compared to AUSM-family**
    - **Controlled numerical dissipation makes pressure flux insufficient, leading to inaccurate results (In particular, around mach number 0.5)**

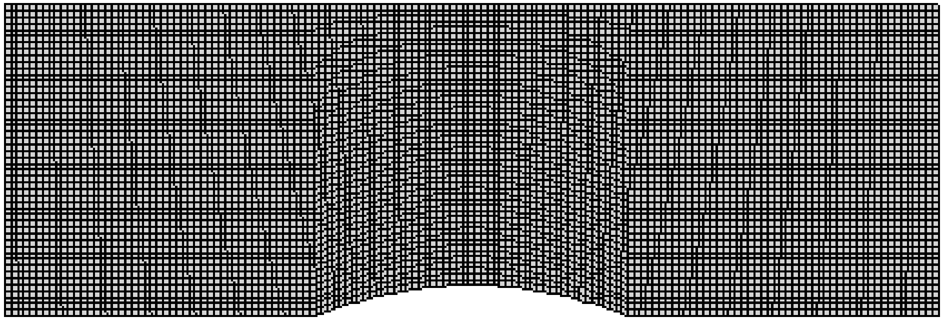
$$p_{\frac{1}{2}} = \frac{p_L + p_R}{2} + \left(P_L^+ p_L + P_R^- p_R - \frac{p_L + p_R}{2}\right) \min(1, M)$$

  - **Difficulties in applying to unstructured grid**
    - **Baseline scheme(M-AUSMPW+) uses superbee value as upper/lower bound to satisfy TVD condition**
    - **Since TVD limiter cannot be applied to unstructured grid, LM-AUSMPW+ scheme need to be fixed for unstructured grid**

$$\Phi_{L,\frac{1}{2}} = \Phi_L + \frac{max(0, (\Phi_R - \Phi_L)(\Phi_{L,sup} - \Phi_L))}{(\Phi_R - \Phi_L)\left|\Phi_{L,sup} - \Phi_L\right|} min\left[a\frac{|\Phi_R - \Phi_L|}{2}, \left|\Phi_{L,sup} - \Phi_L\right|\right]$$

Accuracy problem

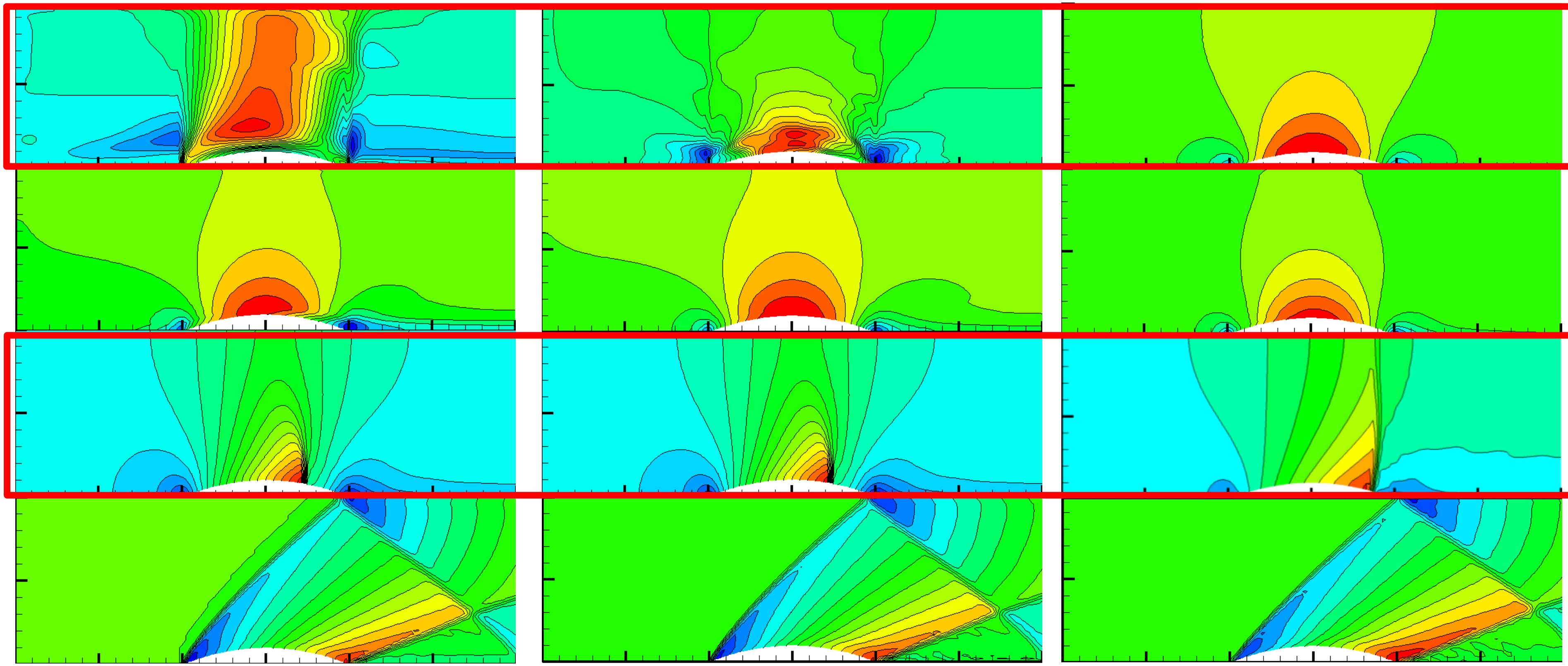
- **Inviscid flow past a circular bump**
  - **10% circular bump**
  - **[0,3] X [0,1] : 150 X 50 mesh is used**
  - **Freestream mach number : 0.001, 0.1, 0.675, 2.0**
    - **Governing equations : Euler equations**
    - **Spatial discretization : AUSMPW+ / M-AUSMPW+ / LM-AUSMPW+**
    - **Reconstruction : 3rd order MLP limiter**
    - **Time integration : LU-SGS with local time stepping**



Structured mesh of bump problem

Accuracy problem

- **Inviscid flow past a circular bump**
  - AUSMPW+ / M-AUSMPW+ / LM-AUSMPW+



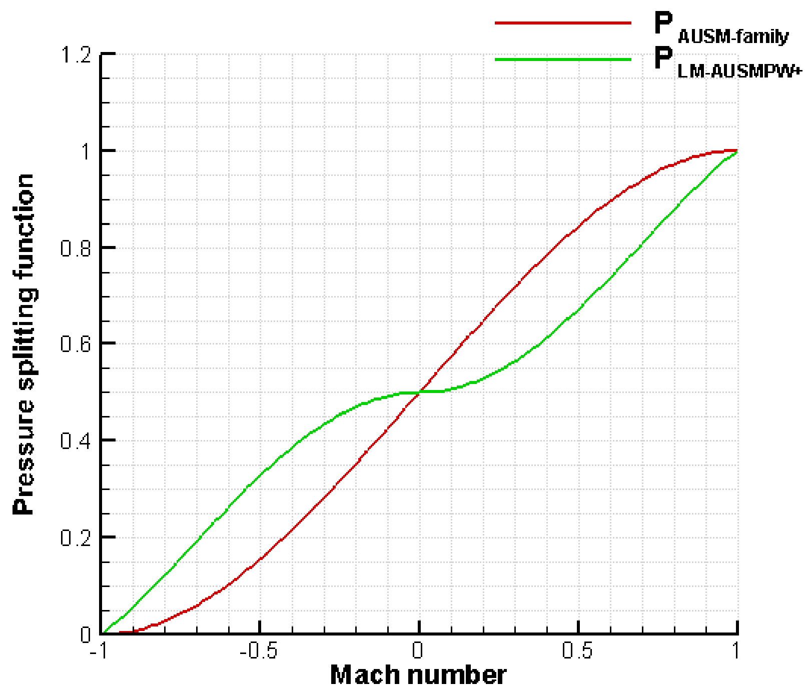
Mach number contours around 10% bump with Mach number 0.001, 0.1, 0.675, 2.0

Accuracy problem

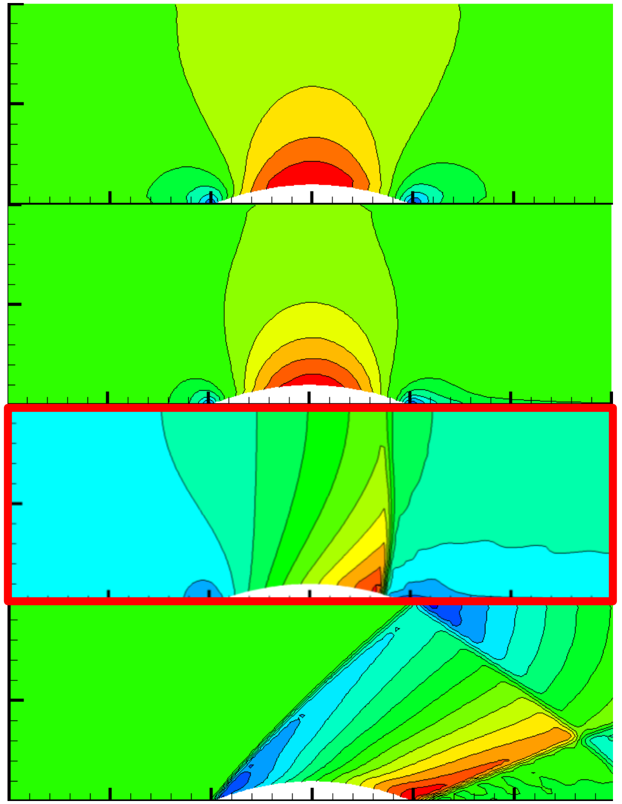
- **Inviscid flow past a circular bump**
  - **Unphysical solution at transonic region due to the low pressure flux**

$$P_{\frac{1}{2},AUSM-family} = P_L^+ p_L + P_R^- p_R$$

$$P_{\frac{1}{2},LM-AUSMPW+} = \frac{p_L + p_R}{2} + D_p \min(1, M)$$
$$= [0.5 + \{P_L^+ - 0.5\}\min(1, M)]p_L + [0.5 + \{P_R^- - 0.5\}\min(1, M)]p_R$$



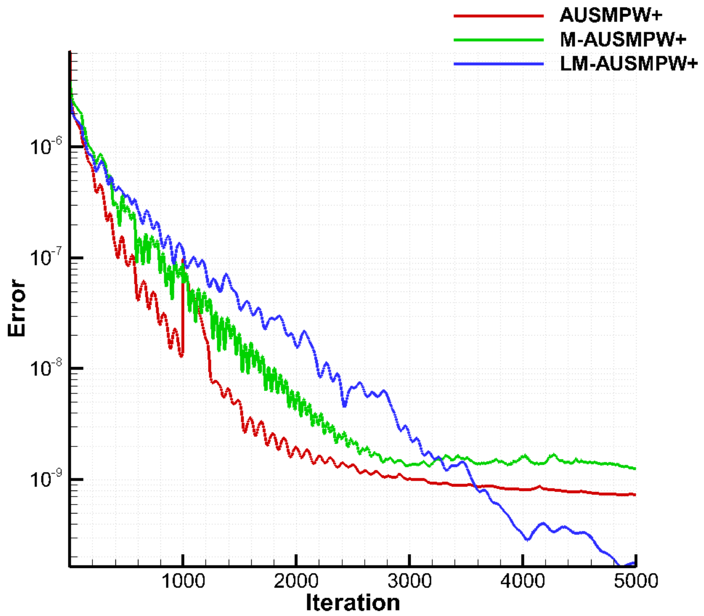
Pressure splitting function (P_L^+)



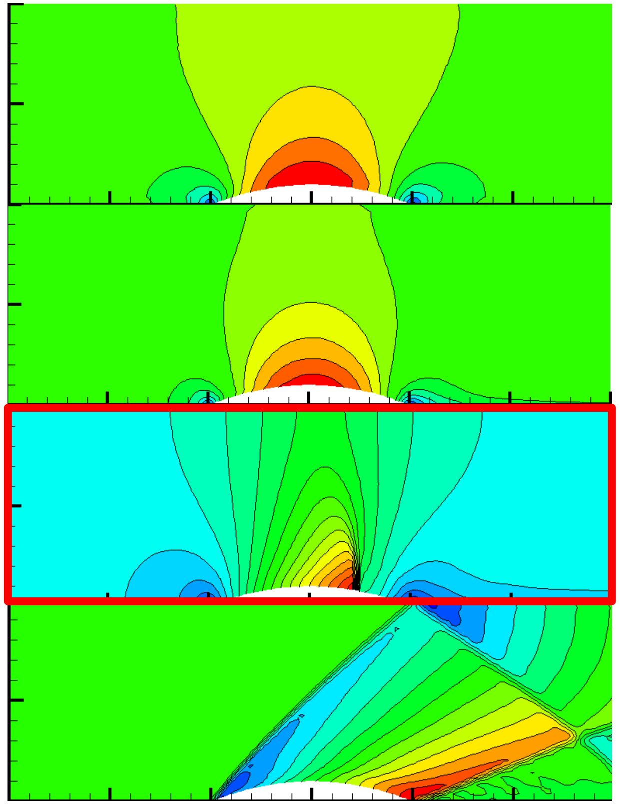
LM-AUSMPW+ results

Accuracy problem

- Inviscid flow past a circular bump
  - Blending two functions
    - Pressure splitting function : Continuous & Differentiable
    - Convergence & Accuracy



Error history (Mach number 0.001)



New LM-AUSMPW+ results

Difficulties in applying to unstructured grid

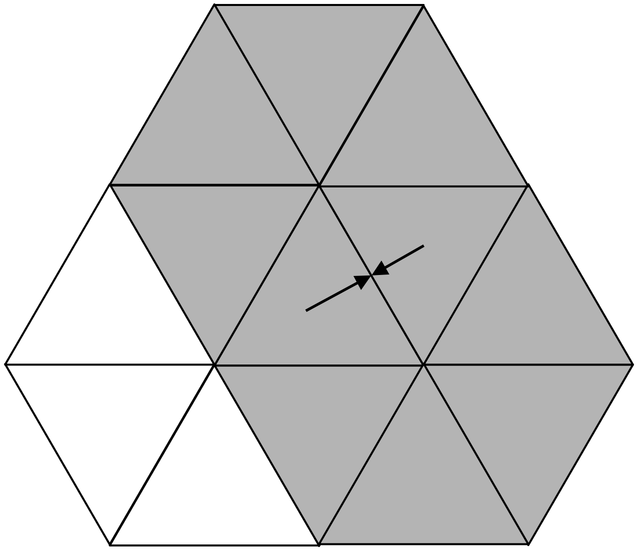
- **Superbee value is replaced by MLP on unstructured grid**

$$q_{jk,\frac{1}{2}} = q_{jk} + \frac{\max\left(0,(q_{kj}-q_{jk})(q_{jk,MLPu1}-q_{jk})\right)}{(q_{kj}-q_{jk})|q_{jk,MLPu1}-q_{jk}|} \times min\left[a\frac{|q_{kj}-q_{jk}|}{2}, |q_{jk,MLPu1} - q_{jk}|\right]$$

$$q_{jk,MLPu1} = q_0 + \phi_{MLP}\nabla q_0 \cdot r_{jk}$$

$$\phi_{MLP} = \begin{cases} \phi(r_{jk}^{max}) & if \nabla q_0 \cdot r_{jk} > 0 \\ \phi(r_{jk}^{min}) & if \nabla q_0 \cdot r_{jk} > 0 \\ 1 & otherwise \end{cases}$$

$$\phi = \min(1, r^{\max\, or\, min})$$



Difficulties in applying to unstructured grid

Stationary contact discontinuity

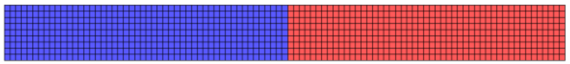
- Initial condition

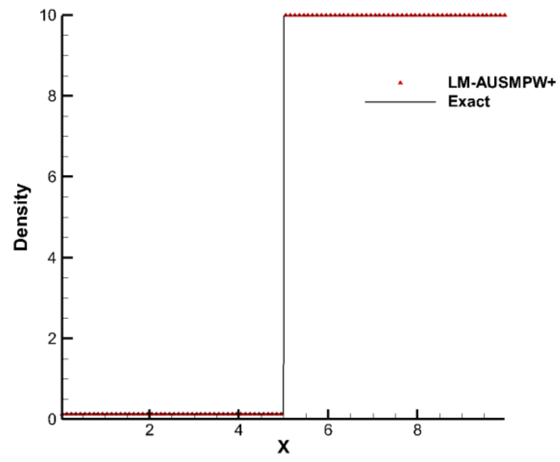
$$(\rho_L, u_L, p_L) = (0.125, 0, 1), \qquad (\rho_R, u_R, p_R) = (10, 0, 1)$$

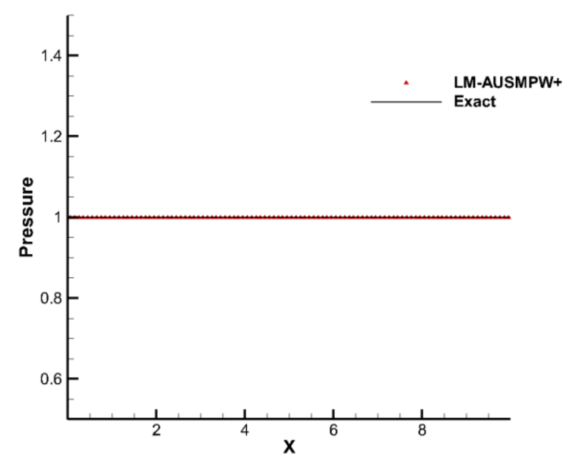
- Discontinuity in density can be preserved by zero mass/pressure flux.
- *Discontinuity line is preserved although numerical dissipation is added.*

$$\bar{M}_L^+ = M_L^+ + M_R^- \cdot [(1-w)(1+f_R) - f_L] = 0$$
$$\bar{M}_R^- = M_R^- \cdot w(1+f_R) = 0$$

$$p_{\frac{1}{2}} = p_L = p_R$$







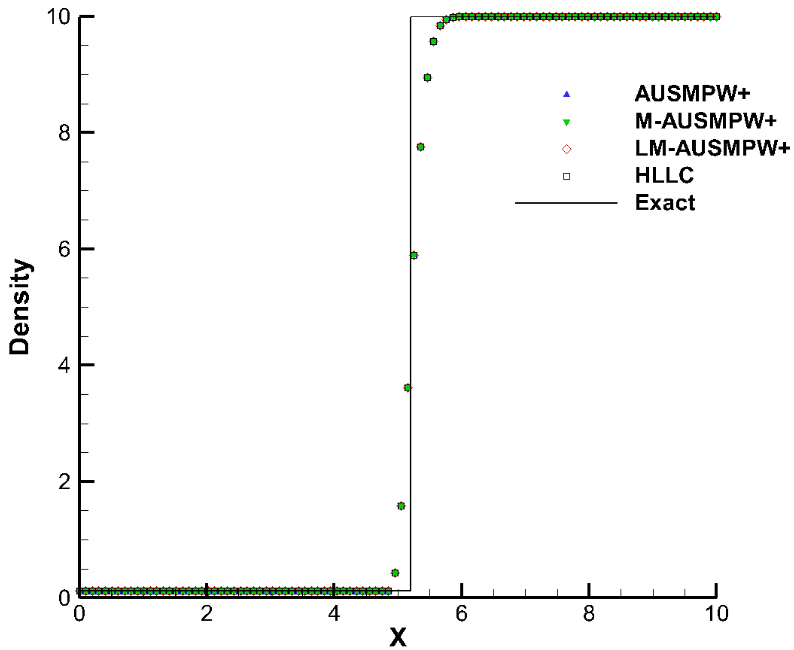
Difficulties in applying to unstructured grid

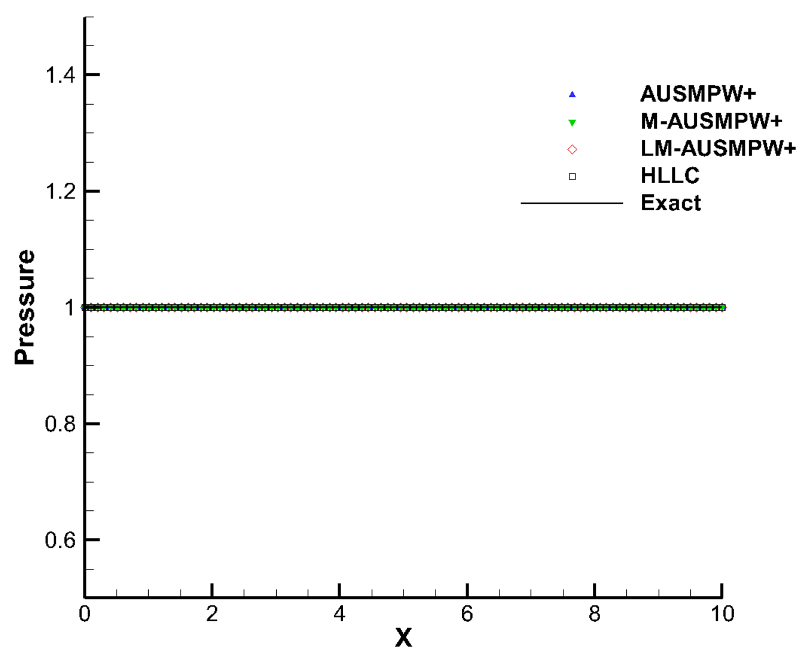
Moving contact discontinuity

- Initial condition

$$(\rho_L, u_L, p_L) = (0.125, 0.1125, 1), \qquad (\rho_R, u_R, p_R) = (10, 0.1125, 1)$$

- TVD Runge − Kutta 3^{th} order method with $\Delta t = 0.005$ until $t = 2.5\text{sec}$
- [0,10] with 100 points, 1^{st} order reconstruction $\boldsymbol{without\ re-evaluation\ process}$





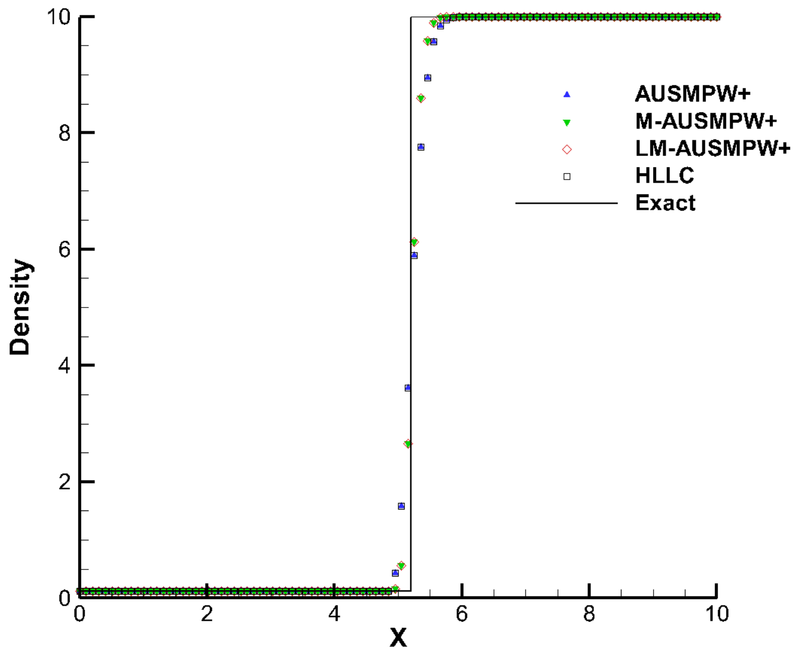
Difficulties in applying to unstructured grid

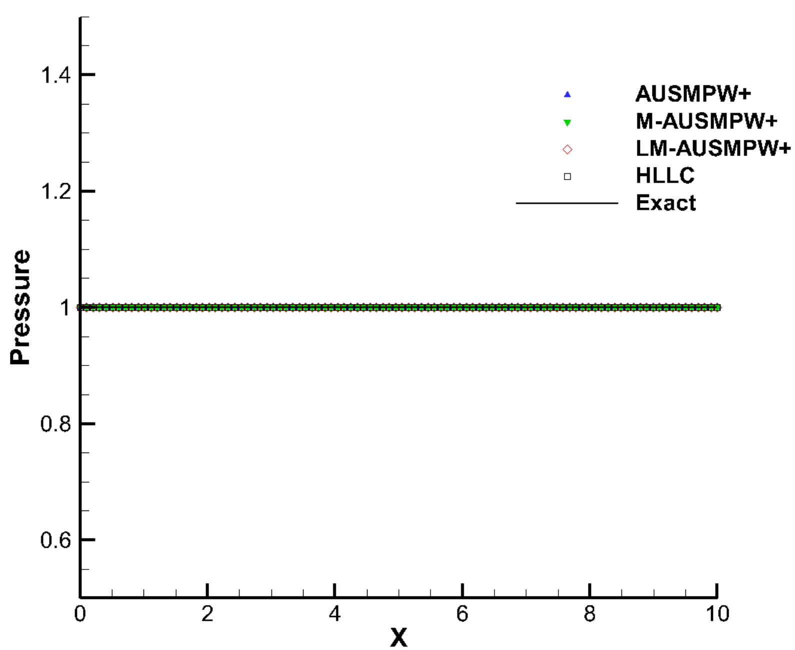
Moving contact discontinuity

- Initial condition

$$(\rho_L, u_L, p_L) = (0.125, 0.1125, 1), \qquad (\rho_R, u_R, p_R) = (10, 0.1125, 1)$$

- TVD Runge – Kutta 3^{th} order method with $\Delta t = 0.005$ until $t = 2.5$sec
- [0,10] with 100 points, 1^{st} order reconst*ruction* $\boldsymbol{with\ re - evaluation\ process}$





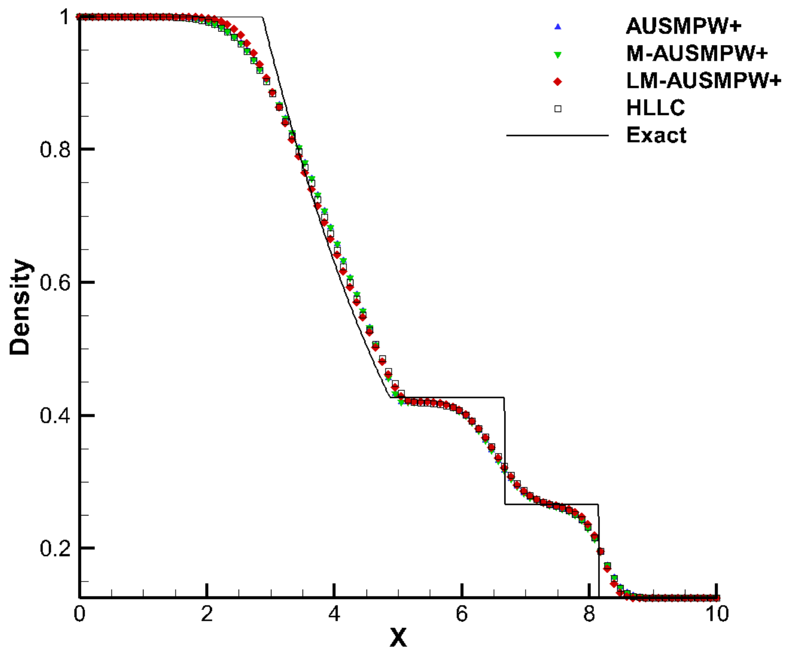
Difficulties in applying to unstructured grid

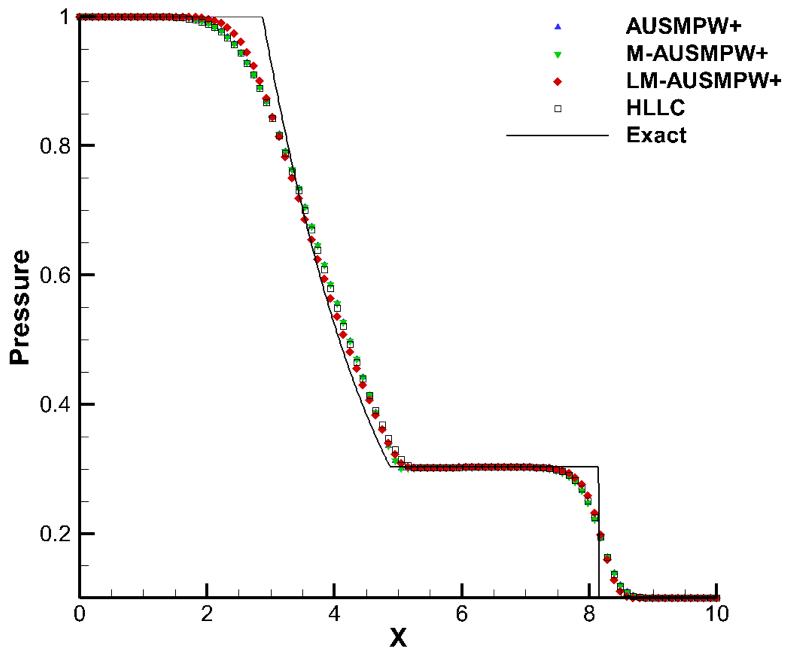
Sod problem

- Initial condition

$$(\rho_L, u_L, p_L) = (1, 0, 1), \qquad (\rho_R, u_R, p_R) = (0.125, 0, 0.1)$$

- TVD Runge – Kutta 3^{th} order method with $\Delta t = 0.005$ until $t = 1.8\text{sec}$
- [0,10] with 100 points, 1^{st} order reconstruction ***without re – evaluation process***





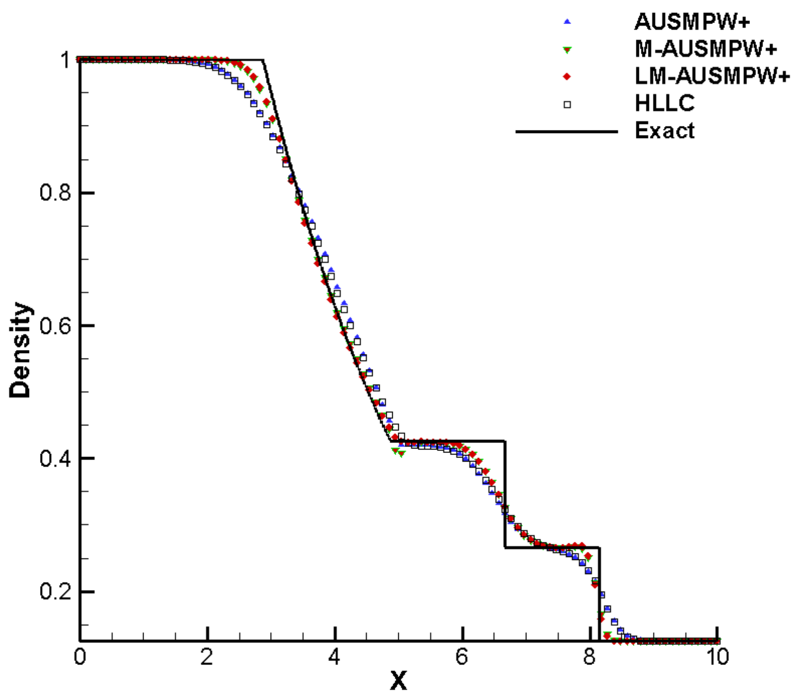
Difficulties in applying to unstructured grid

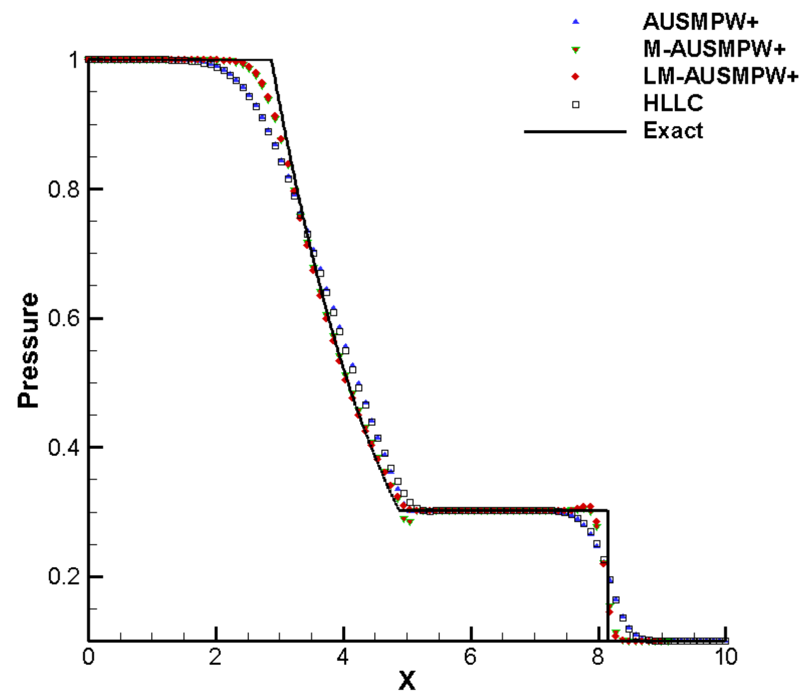
Sod problem

- Initial condition

$$(\rho_L, u_L, p_L) = (1, 0, 1), \qquad (\rho_R, u_R, p_R) = (0.125, 0, 0.1)$$

- TVD Runge − Kutta 3^{th} order method with $\Delta t = 0.005$ until $t = 1.8\text{sec}$
- [0,10] with 100 points, 1^{st} order reconstruction ***with re − evaluation process***





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4. Conclusion

Conclusion

- **Summary**
  - **LM-AUSMPW+ : extension of M-AUSMPW+ scheme for all-speed flow**
    - **Accuracy problem at the transonic region is cured**
    - **Pressure splitting function is newly defined to be continuous and differentiable**

    - **LM-AUSMPW+ scheme is extended to the unstructured grid**
    - **Superbee limiter value is replaced by MLP-u1 value**

Thank you for listening

Receding shock

- **Supersonic inverse flow problem**
  - **Initial condition**

$$(\rho_L, u_L, p_L) = (1, -2, 0.4), \qquad (\rho_R, u_R, p_R) = (1, 2, 0.4)$$

  - ***Asdf***

Receding shock

- **Subsonic inverse flow problem**
  - **Initial condition**

$$(\rho_L, u_L, p_L) = (1, -0.5, 0.4), \qquad (\rho_R, u_R, p_R) = (1, 0.5, 0.4)$$

  - ***Asdf***

Riemann problem 5

- **Shock entropy wave interaction**
  - Initial condition

$$(\rho_L, u_L, p_L) = \left(\frac{27}{7}, \frac{4\sqrt{35}}{9}, \frac{31}{3}\right), \qquad (\rho_R, u_R, p_R) = (1 + \epsilon sin(kx), 0, 1)$$

  - $\mathrm{TVD\ Runge-Kutta\ 3^{th}\ order\ method\ with\ \Delta t = 0.005\ until\ t = 1.8sec}$
  - $\mathrm{[0{,}10]\ with\ 100\ points, 1^{st}\ order\ recons}truction\ \boldsymbol{without\ re-evaluation\ process}$

Riemann problem 5

- **Shock entropy wave interaction**
  - Initial condition

$$(\rho_L, u_L, p_L) = \left(\frac{27}{7}, \frac{4\sqrt{35}}{9}, \frac{31}{3}\right), \qquad (\rho_R, u_R, p_R) = (1 + \epsilon sin(kx), 0, 1)$$

  - $\mathrm{TVD\ Runge - Kutta\ 3^{th}\ order\ method\ with\ \Delta t = 0.005\ until\ t = 1.8sec}$
  - $\mathrm{[0,10]\ with\ 100\ points, 1^{st}\ order\ recons}truction\ \boldsymbol{with\ re - evaluation\ process}$

Riemann problem 6

◆ Receding supersonic expansion

- Initial condition

$$(\rho_L, u_L, p_L) = (1, -2, 0.4), \qquad (\rho_R, u_R, p_R) = (1, 2, 0.4)$$

- TVD Runge − Kutta 3^{th} order method with $\Delta t = 0.005$ until $t = 1.8\mathrm{sec}$
- [0,10] with 100 points, 1^{st} order reconstruction $\boldsymbol{with\ re-evaluation\ process}$

2D bump

- **Result**