LM-AUSMPW+ scheme for all-speed on unstructured grid

Taeyoon Kung, Kyuhong Kim

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1. **Introduction**

Introduction

◆ **Characteristics which flow solver needs to have**

Robustness & Accuracy

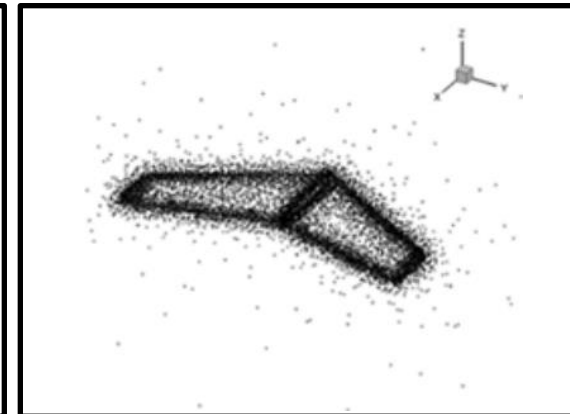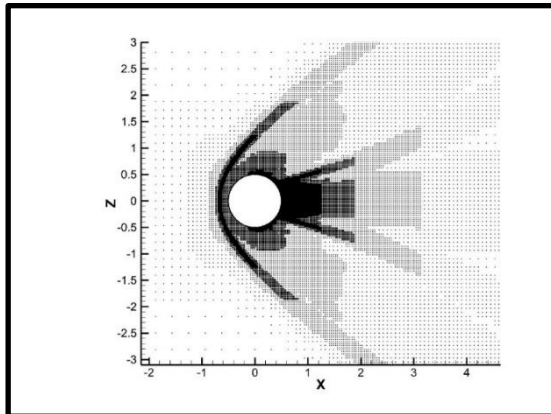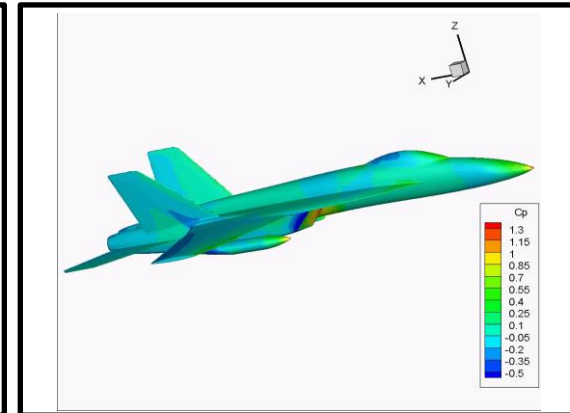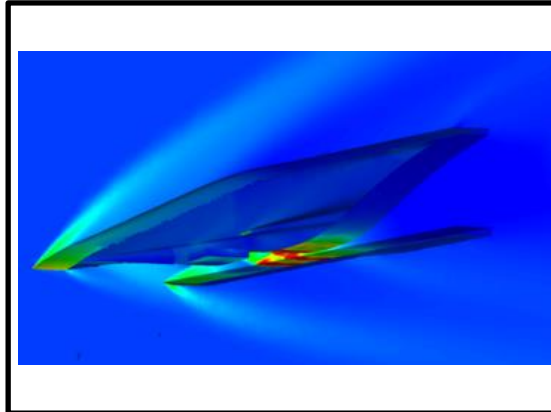Multi-physics

Turbulent flow
Plasma
Ablation
All-speed flow
Moving object analysis

Efficiency

Mesh optimization
Fast convergence

Introduction

◆ **All-speed flow**

- **The mathematical properties of the equations vary depending on the flow velocity range.**

- **It is common for users to use single method for the whole flow field.**

- **Most flux schemes which successfully solve the accuracy problem at supersonic speed have difficulties in obtaining low Mach number flow solution.**
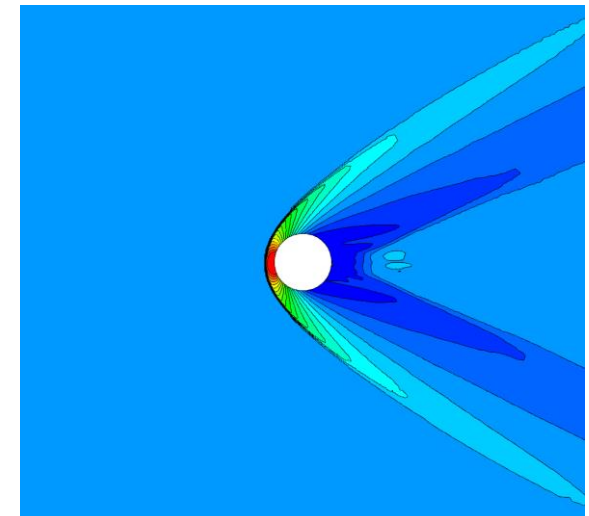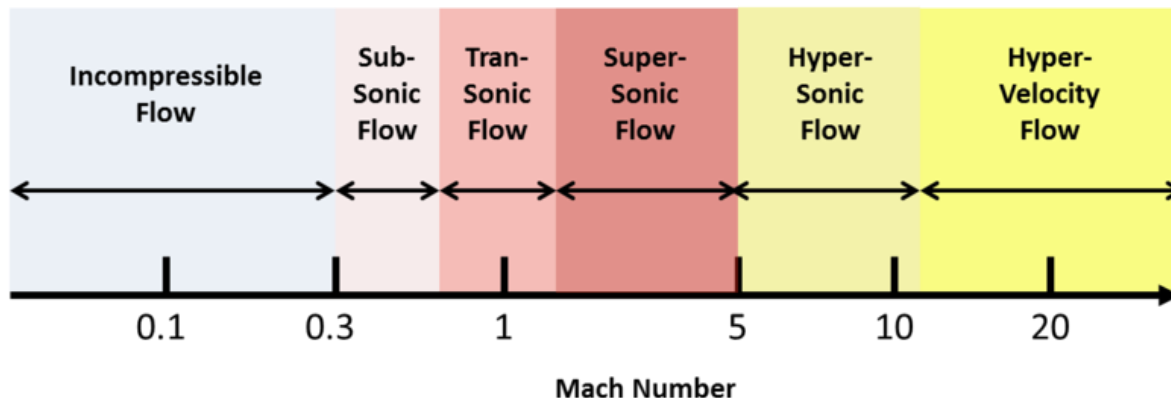
Mach Number Flow Regimes

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2 **Previous study**

- Governing equations

- Baseline scheme

- LM-AUSMPW+

Governing equations

◆ **Euler equations**

■ **Conservative form is used for compressible flow**

$$\frac{\partial q}{\partial t} + \nabla \cdot f_c(q) = 0$$

$$q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e_t \end{bmatrix}, \qquad f_c(q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uH \end{bmatrix} \hat{\imath} + \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vH \end{bmatrix} \hat{\jmath}$$

■ **Nondimensionalization**

$$\rho^* = \frac{\rho}{\rho_\infty}, u^* = \frac{u}{u_\infty}, v^* = \frac{v}{v_\infty}, x^* = \frac{x}{l_\infty},$$

$$y^* = \frac{y}{l_\infty}, p^* = \frac{p}{\rho_\infty c_\infty^2}, e^* = \frac{e}{c_\infty^2}, t^* = \frac{t}{u_\infty/l_\infty}$$

$$\frac{\partial \rho^*}{\partial t^*} + \nabla(\rho^* u^*) = 0$$

$$\frac{\partial}{\partial t^*} \rho^* u^* + \nabla(\rho^* u^* u^*) = -\frac{1}{M_\infty^2} \nabla p^*$$

$$\frac{\partial}{\partial t^*} \rho^* e^* + \nabla(\rho^* e^* u^* + p^* u^*) = 0$$

Baseline scheme: AUSMPW+/M-AUSMPW+

◆ **AUSMPW+**

■ **Control convection property by considering both left and right states across strong shock ▶ numerical oscillation/overshoot is cured**

$$F_{\frac{1}{2}} = \bar{M}_L^+ c_{\frac{1}{2}} \Phi_L + \bar{M}_R^- c_{\frac{1}{2}} \Phi_R + P_L^+ p_L + P_R^- p_R$$

$if \ M_L^+ + M_R^- \geq 0,$
$\bar{M}_L^+ = M_L^+ + M_R^- \cdot [(1-w)(1+f_R) - f_L]$
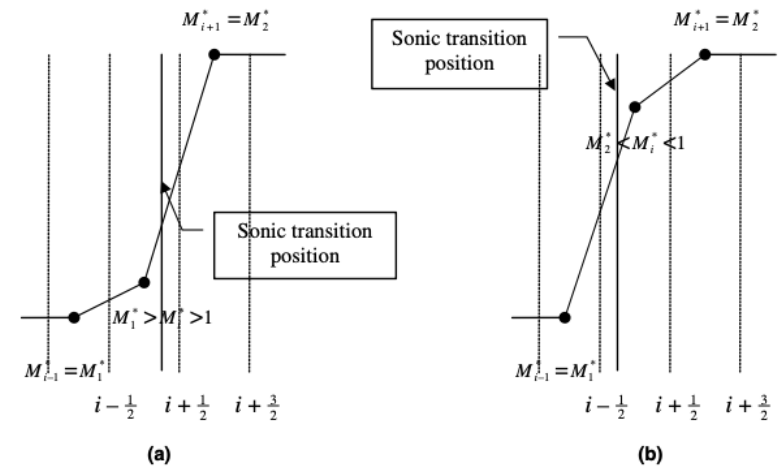$\bar{M}_R^- = M_R^- \cdot w(1+f_R)$

$if \ M_L^+ + M_R^- < 0,$
$\bar{M}_L^+ = M_L^+ \cdot w(1+f_L)$
$\bar{M}_R^- = M_R^- + M_L^+ \cdot [(1-w)(1+f_L) - f_R]$

$f_{L,R} = \left(\dfrac{p_{L,R}}{p_s} - 1\right) \min(1, \dfrac{\min(p_{1,L}, p_{1,R}, p_{2,L}, p_{2,R})}{\min(p_L, p_R)})$

$w = 1 - \min\left(\dfrac{p_R}{p_L}, \dfrac{p_L}{p_R}\right)^3$

(a)   (b)

Baseline scheme: AUSMPW+/M-AUSMPW+

◆ **M-AUSMPW+**

$$F_{\frac{1}{2}} = \bar{M}_L^+ c_{\frac{1}{2}} \Phi_{L,\frac{1}{2}} + \bar{M}_R^- c_{\frac{1}{2}} \Phi_{R,\frac{1}{2}} + P_L^+ p_{L,\frac{1}{2}} + P_R^- p_{R,\frac{1}{2}}$$

- **R1) The region of continuity can be distinguished from discontinuity**
- **R2) Monotonic condition should be satisfied**
- **R3) Convective quantity should maintain upwind characteristic in supersonic flow**

$$\Phi_{L,\frac{1}{2}} = \Phi_L + \frac{max(0,(\Phi_R - \Phi_L)(\Phi_{L,sup} - \Phi_L))}{(\Phi_R - \Phi_L)|\Phi_{L,sup} - \Phi_L|} min\left[a\frac{|\Phi_R - \Phi_L|}{2}, |\Phi_{L,sup} - \Phi_L|\right]$$

$$\Phi_{R,\frac{1}{2}} = \Phi_R + \frac{max(0,(\Phi_L - \Phi_R)(\Phi_{R,sup} - \Phi_R))}{(\Phi_L - \Phi_R)|\Phi_{R,sup} - \Phi_R|} min\left[a\frac{|\Phi_L - \Phi_R|}{2}, |\Phi_{R,sup} - \Phi_R|\right]$$

$$a = 1 - min\left(1, max(|M_L|, |M_R|)\right)^2$$

- **Modification of pressure splitting function which is aimed to improve accuracy in steady shock discontinuity**

$$If\ M_i^* > 1, M_{i+1}^* < 1\ and\ 0 < M_i^* M_{i+1}^* < 1$$

$$P_{i+1}^- = \max\left(0, \min\left(0.5, 1 - \frac{\rho_i U_i(U_i - U_{i+1}) + p_i}{p_{i+1}}\right)\right)$$

$$If\ M_i^* > -1, M_{i+1}^* < -1\ and\ 0 < M_i^* M_{i+1}^* < 1$$

$$P_i^+ = \max\left(0, \min\left(0.5, 1 - \frac{\rho_{i+1} U_{i+1}(U_{i+1} - U_i) + p_{i+1}}{p_i}\right)\right)$$

LM-AUSMPW+ scheme

◆ **New version of M-AUSMPW+ scheme for low-speed accuracy**

   ■ **Manipulation of artificial dissipation based on the asymptotic analysis**

Governing equations

$$p^*(x,t) = p_0^*(t) + M_\infty^2 p_2^*(x,t) + \cdots \ - \ satisfied?$$

AUSMPW+

1) *Order of* $1/M_\infty^2$
$$p_{0,i+1,j}^* - p_{0,i-1,j}^* = 0$$
$$p_{0,i,j+1}^* - p_{0,i,j-1}^* = 0$$

2) *Order of* $1/M_\infty$
$$\frac{1}{2}\sum_l p_{1,l}(\overrightarrow{n_{il}})_x - \frac{3}{4}\sum_l \left[\frac{u_{0,l}\cdot\overrightarrow{n_{x,il}}}{c_{ij}}p_{0,l}\right] = 0$$
$$\frac{1}{2}\sum_l p_{1,l}(\overrightarrow{n_{il}})_y - \frac{3}{4}\sum_l \left[\frac{u_{0,l}\cdot\overrightarrow{n_{y,il}}}{c_{ij}}p_{0,l}\right] = 0$$

M-AUSMPW+

1) *Order of* $1/M_\infty^2$
$$p_{0,i+1,j}^* - p_{0,i-1,j}^* = 0$$
$$p_{0,i,j+1}^* - p_{0,i,j-1}^* = 0$$

2) *Order of* $1/M_\infty$
$$p_{1,i+1,j}^* - p_{1,i-1,j}^* = 0$$
$$p_{1,i,j+1}^* - p_{1,i,j-1}^* = 0$$

LM-AUSMPW+ scheme

◆ **New version of M-AUSMPW+ scheme for low-speed accuracy**

■ **Convective term of LM-AUSMPW should satisfy following requirements**

- C1) Asymptotic characteristics of governing equations should be satisfied in additional numerical dissipation.
- C2) Supersonic characteristics of baseline scheme should be maintained.
- C3) The size of numerical dissipation is determined according to Mach number region.

■ **The order of numerical dissipation is manipulated to satisfy asymptotic behavior of governing equations at low mach number region**

■ **Additional numerical dissipation is added to pressure term tied with momentum equations to cure the instability due to zero numerical dissipation**

$$p_{\frac{1}{2}} = \frac{p_L + p_R}{2} + \left(P_L^+ p_L + P_R^- p_R - \frac{p_L + p_R}{2} \right) \min(1, M)$$
$$+ P_L^+ P_R^- M^2 \left(1 - min(1, max(|M_L|, |M_R|))^2 \right) (p_R - p_L)$$
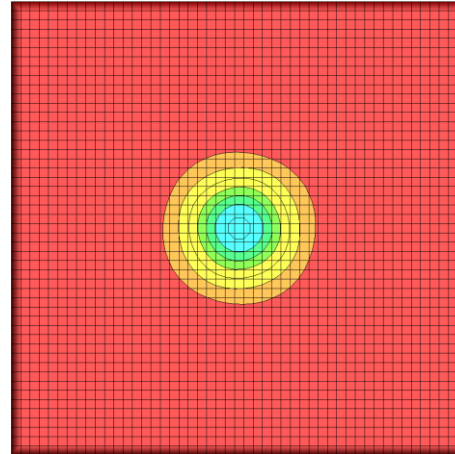
LM-AUSMPW+ scheme

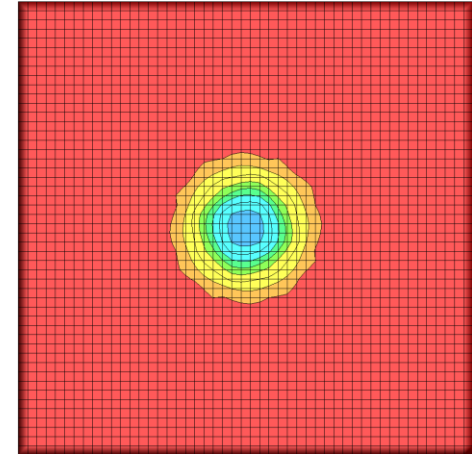◆ **New version of M-AUSMPW+ scheme for low-speed accuracy**

■ *Low numerical dissipation in the region where Mach number is under* 1
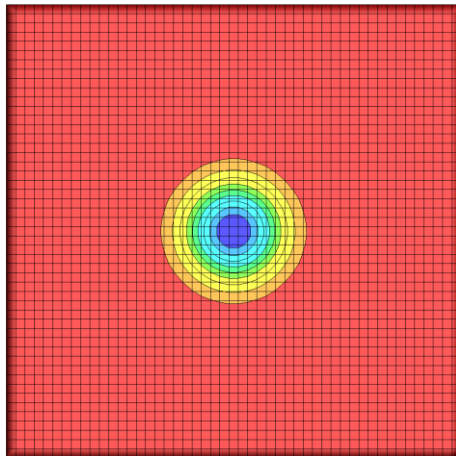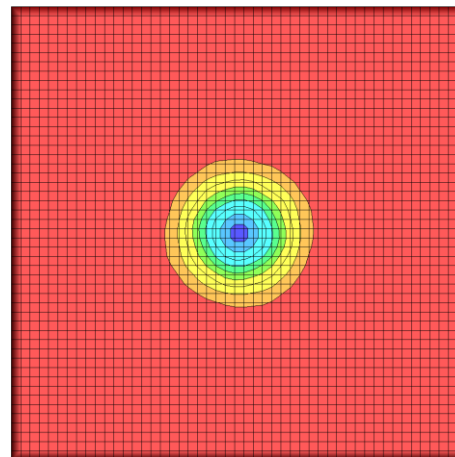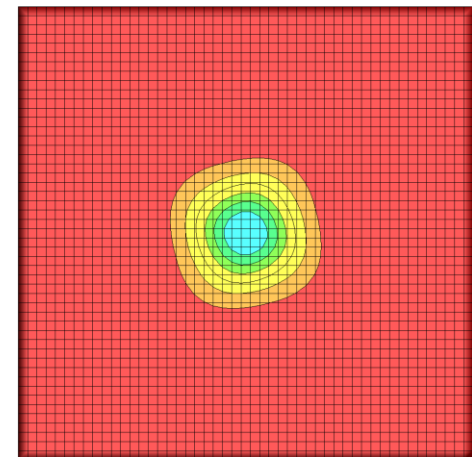

Initial

AUSMPW +

M − AUSMPW +

LM − AUSMPW +

HLLC

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3. **Improvement on scheme**

Limitation of LM-AUSMPW+ scheme

◆ **LM-AUSMPW+ of previous study has 2 limitations**

■ **Accuracy problem**

- ♦ **Artificial viscosity is always decreased in LM-AUSMPW+ scheme, compared to AUSM-family**

- ♦ **Controlled numerical dissipation makes pressure flux insufficient, leading to inaccurate results (In particular, around mach number 0.5)**

$$p_{\frac{1}{2}} = \frac{p_L + p_R}{2} + \left(P_L^+ p_L + P_R^- p_R - \frac{p_L + p_R}{2} \right) \min(1, M)$$

■ **Difficulties in applying to unstructured grid**

- ♦ **Baseline scheme(M-AUSMPW+) uses superbee value as upper/lower bound to satisfy TVD condition**

- ♦ **Since TVD limiter cannot be applied to unstructured grid, LM-AUSMPW+ scheme need to be fixed for unstructured grid**

$$\Phi_{L,\frac{1}{2}} = \Phi_L + \frac{max(0, (\Phi_R - \Phi_L)(\Phi_{L,sup} - \Phi_L))}{(\Phi_R - \Phi_L)|\Phi_{L,sup} - \Phi_L|} min\left[a\frac{|\Phi_R - \Phi_L|}{2}, |\Phi_{L,sup} - \Phi_L| \right]$$

Accuracy problem

◆ **Inviscid flow past a circular bump**

■ **10% circular bump**

■ **[0,3] X [0,1] : 150 X 50 mesh is used**

■ **Freestream mach number : 0.001, 0.1, 0.675, 2.0**

  ◆ **Governing equations : Euler equations**

  ◆ **Spatial discretization : AUSMPW+ / M-AUSMPW+ / LM-AUSMPW+**

  ◆ **Reconstruction : 3rd order MLP limiter**

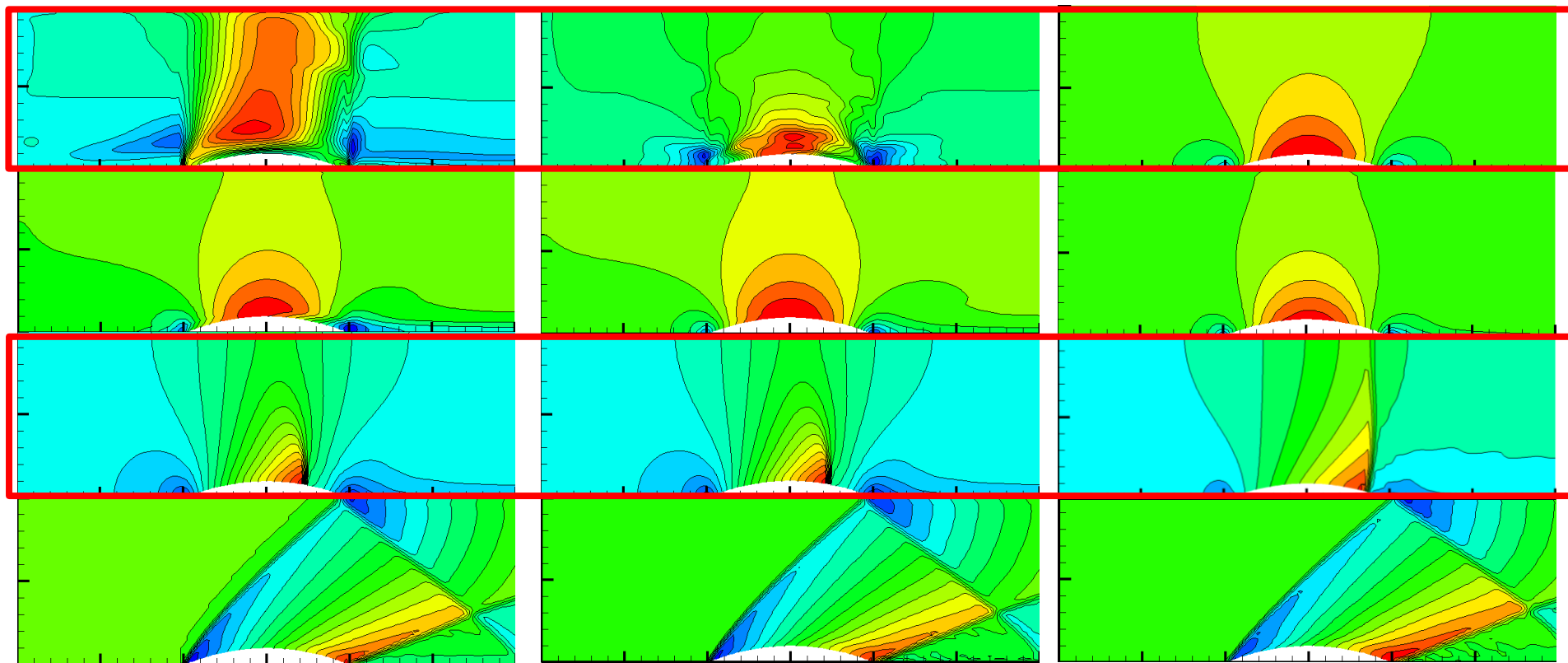 ◆ **Time integration : LU-SGS with local time stepping**
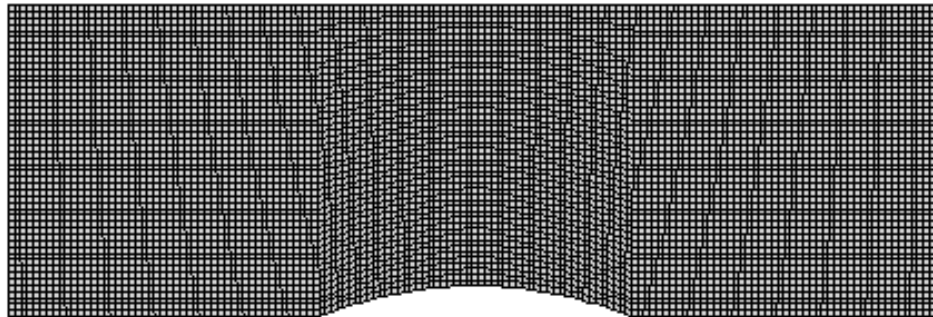
Structured mesh of bump problem

Accuracy problem

◆ **Inviscid flow past a circular bump**

■ **AUSMPW+ / M-AUSMPW+ / LM-AUSMPW+**

Mach number contours around 10% bump with Mach number 0.001, 0.1, 0.675, 2.0
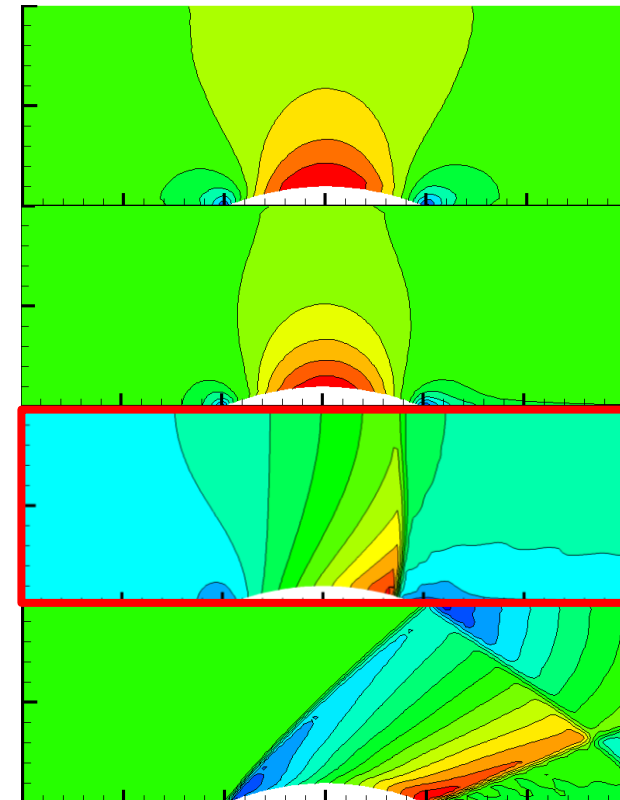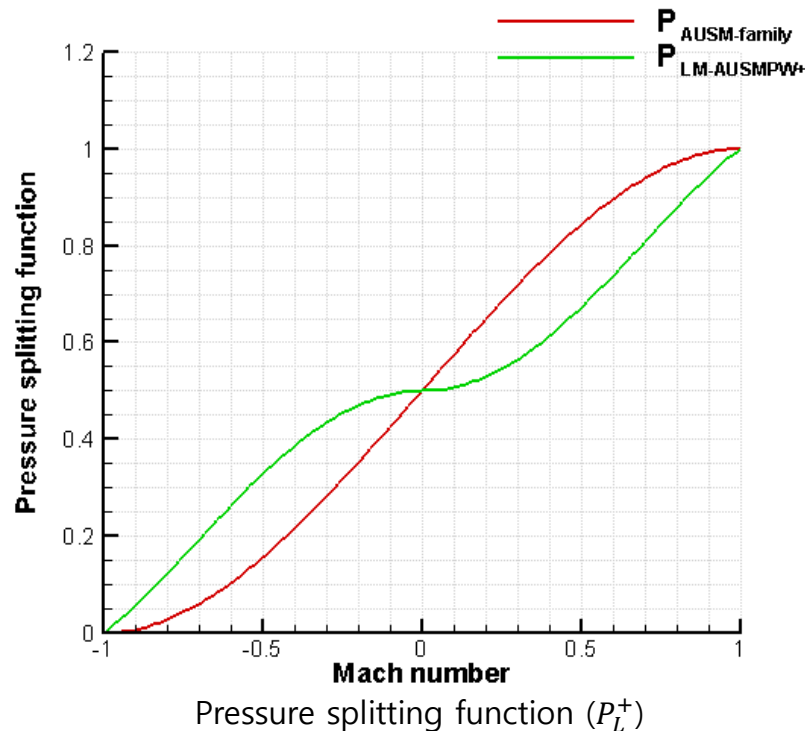
Accuracy problem

◆ **Inviscid flow past a circular bump**

- **Unphysical solution at transonic region due to the low pressure flux**

$$P_{\frac{1}{2}, AUSM-family} = P_L^+ p_L + P_R^- p_R$$

$$P_{\frac{1}{2}, LM-AUSMPW+} = \frac{p_L + p_R}{2} + D_p \min(1, M)$$
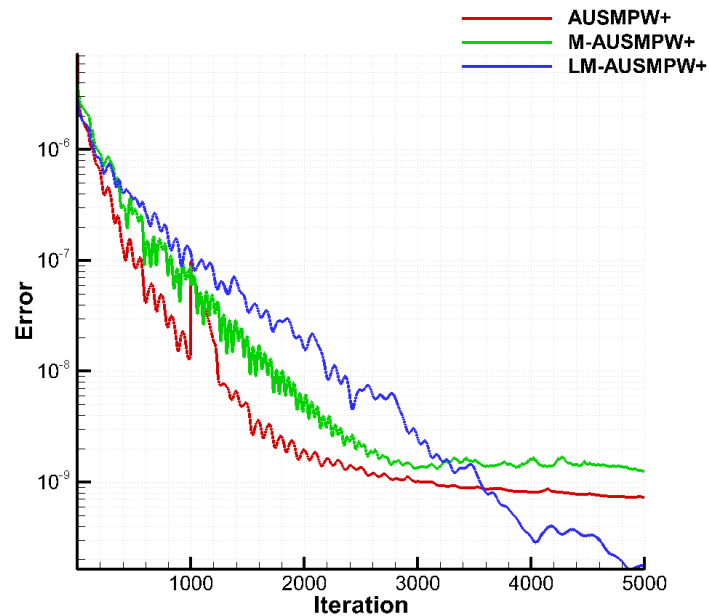$$= [0.5 + \{P_L^+ - 0.5\}\min(1, M)]p_L + [0.5 + \{P_R^- - 0.5\}\min(1, M)]p_R$$

Pressure splitting function (P_L^+)
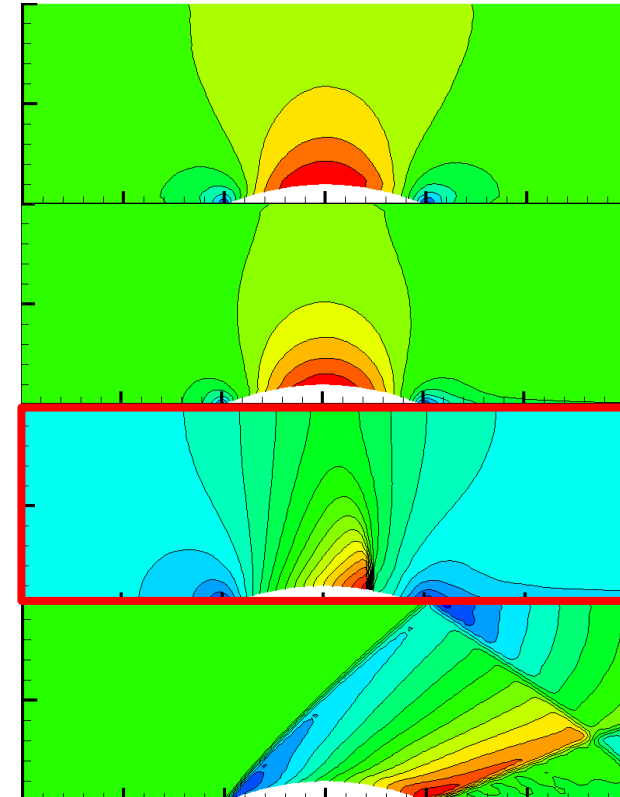
LM-AUSMPW+ results

Accuracy problem

◆ Inviscid flow past a circular bump

- ### Blending two functions
  - Pressure splitting function : Continuous & Differentiable
  - Convergence & Accuracy

Error history (Mach number 0.001)

New LM-AUSMPW+ results
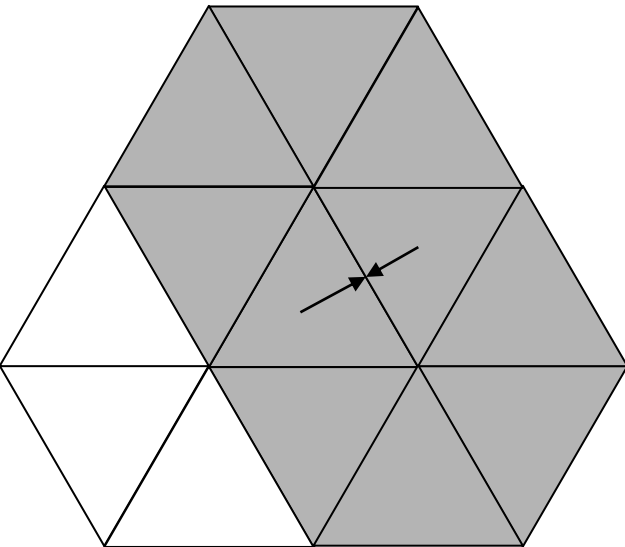
Difficulties in applying to unstructured grid

◆ **Superbee value is replaced by MLP on unstructured grid**

$$q_{jk,\frac{1}{2}} = q_{jk} + \frac{\max\left(0,(q_{kj}-q_{jk})(q_{jk,MLPu1}-q_{jk})\right)}{(q_{kj}-q_{jk})|q_{jk,MLPu1}-q_{jk}|} \times min\left[a\frac{|q_{kj}-q_{jk}|}{2}, |q_{jk,MLPu1}-q_{jk}|\right]$$

$$q_{jk,MLPu1} = q_0 + \phi_{MLP}\nabla q_0 \cdot r_{jk}$$

$$\phi_{MLP} = \begin{cases} \phi\left(r_{jk}^{max}\right) & if \nabla q_0 \cdot r_{jk} > 0 \\ \phi\left(r_{jk}^{min}\right) & if \nabla q_0 \cdot r_{jk} > 0 \\ 1 & otherwise \end{cases}$$

$$\phi = \min(1, r^{\max\ or\ min})$$

Difficulties in applying to unstructured grid

◆ **Stationary contact discontinuity**
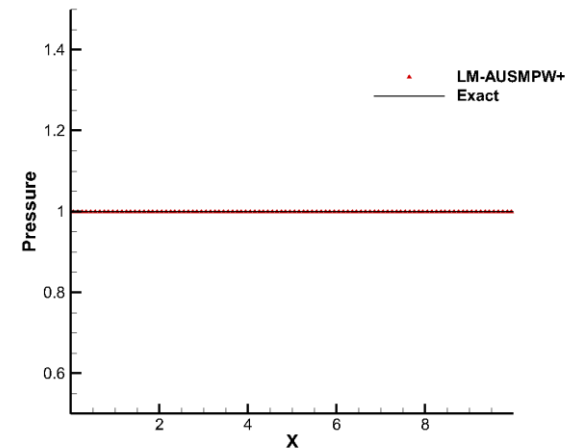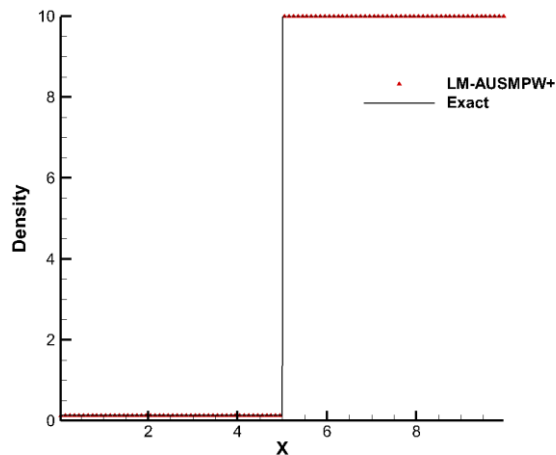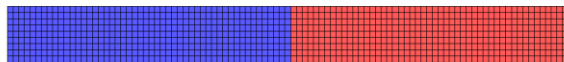
- ■ Initial condition

$$(\rho_L, u_L, p_L) = (0.125, 0, 1), \qquad (\rho_R, u_R, p_R) = (10, 0, 1)$$

- ■ Discontinuity in density can be preserved by zero mass/pressure flux.

- ■ *Discontinuity line is preserved although numerical dissipation is added.*

$$\bar{M}_L^+ = M_L^+ + M_R^- \cdot [(1 - w)(1 + f_R) - f_L] = 0$$
$$\bar{M}_R^- = M_R^- \cdot w(1 + f_R) = 0$$
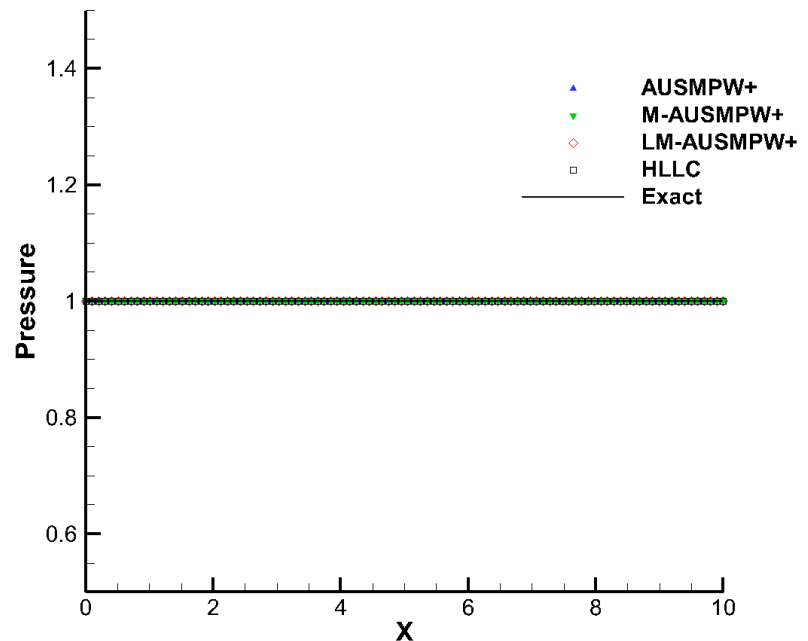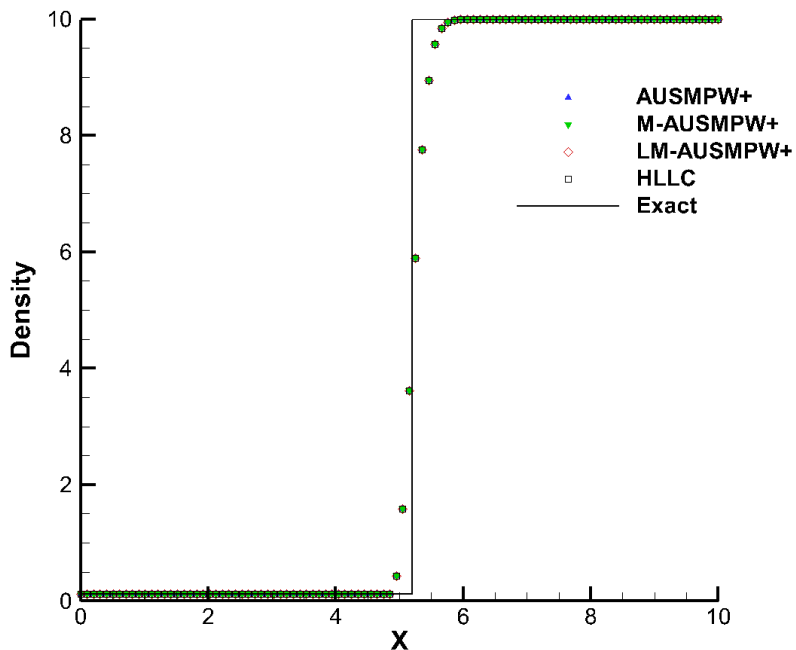
$$p_{\frac{1}{2}} = p_L = p_R$$

Difficulties in applying to unstructured grid

◆ **Moving contact discontinuity**

- ■ Initial condition

$$(\rho_L, u_L, p_L) = (0.125, 0.1125, 1), \qquad (\rho_R, u_R, p_R) = (10, 0.1125, 1)$$

- ■ TVD Runge − Kutta 3$^{\text{th}}$ order method with $\Delta t = 0.005$ until t = 2.5sec

- ■ [0,10] with 100 points, 1$^{\text{st}}$ order recons*truction **without re − evaluation process***
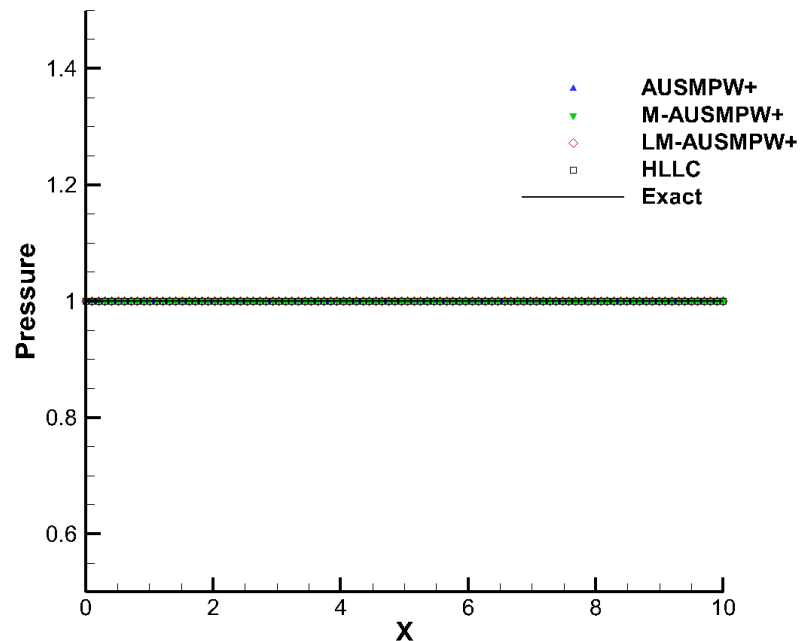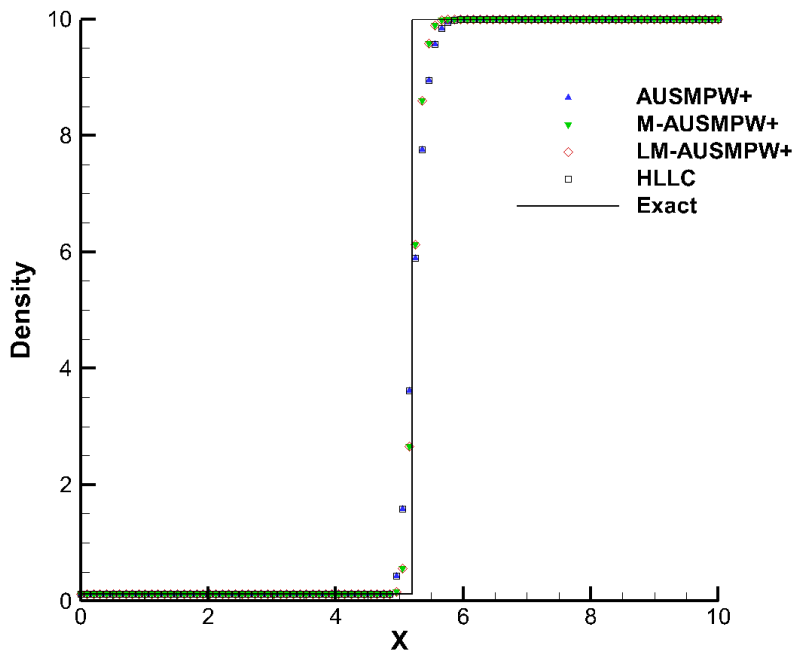
Difficulties in applying to unstructured grid

◆ **Moving contact discontinuity**

■ Initial condition

$$(\rho_L, u_L, p_L) = (0.125, 0.1125, 1), \qquad (\rho_R, u_R, p_R) = (10, 0.1125, 1)$$

■ TVD Runge − Kutta 3th order method with $\Delta t = 0.005$ until t = 2.5sec

■ [0,10] with 100 points, 1st order recons*truction **with re − evaluation process***
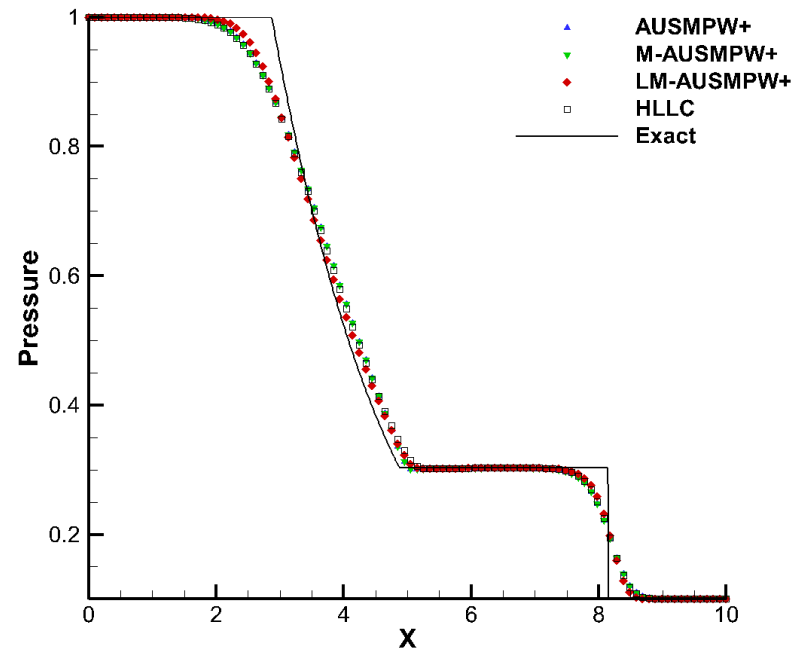
Difficulties in applying to unstructured grid

◆ **Sod problem**

■ Initial condition

$$(\rho_L, u_L, p_L) = (1, 0, 1), \qquad (\rho_R, u_R, p_R) = (0.125, 0, 0.1)$$

■ TVD Runge − Kutta 3th order method with $\Delta t = 0.005$ until t = 1.8sec

■ [0,10] with 100 points, 1st order recons$truction$ ***without re − evaluation process***
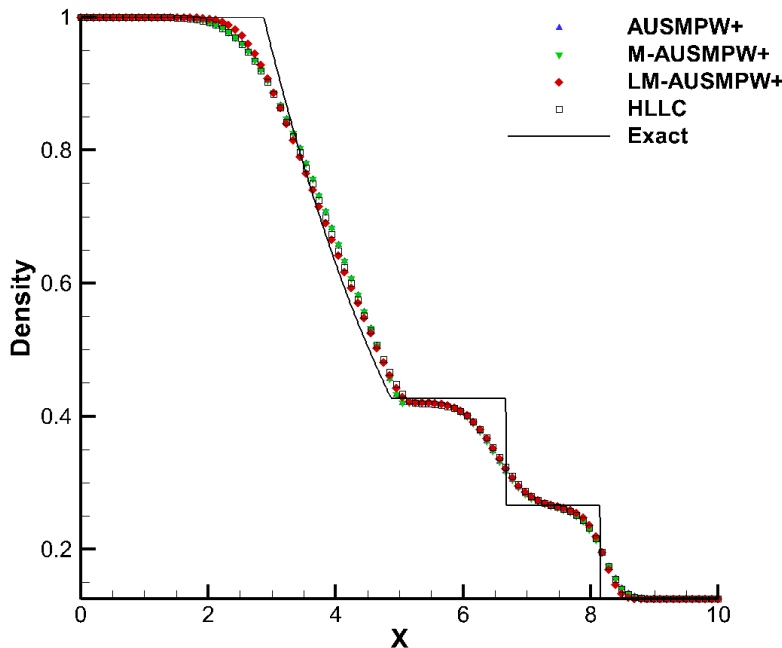
Difficulties in applying to unstructured grid

◆ **Sod problem**

■ Initial condition

$$(\rho_L, u_L, p_L) = (1, 0, 1), \qquad (\rho_R, u_R, p_R) = (0.125, 0, 0.1)$$

■ TVD Runge − Kutta 3$^{\text{th}}$ order method with Δt = 0.005 until t = 1.8sec

■ [0,10] with 100 points, 1$^{\text{st}}$ order recons*truction **with re − evaluation process***
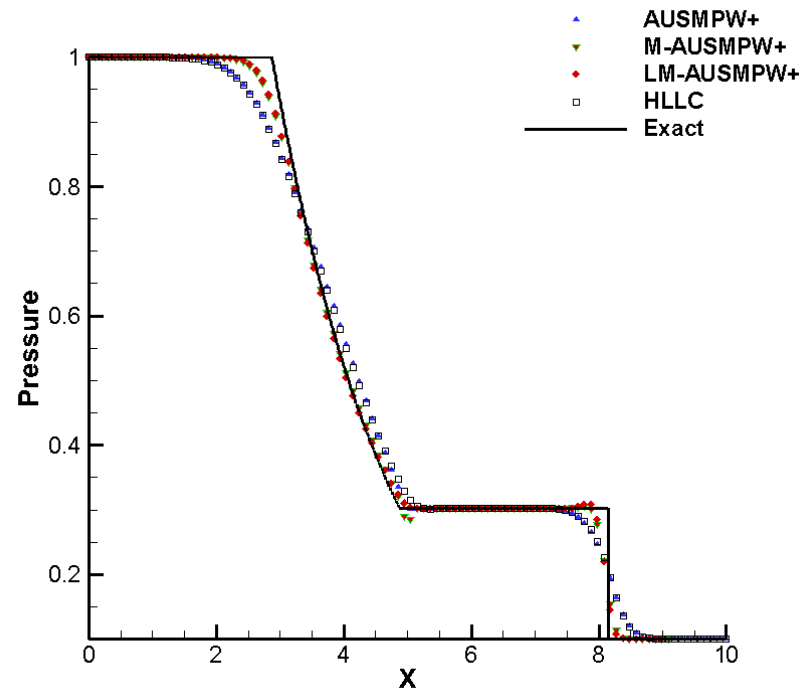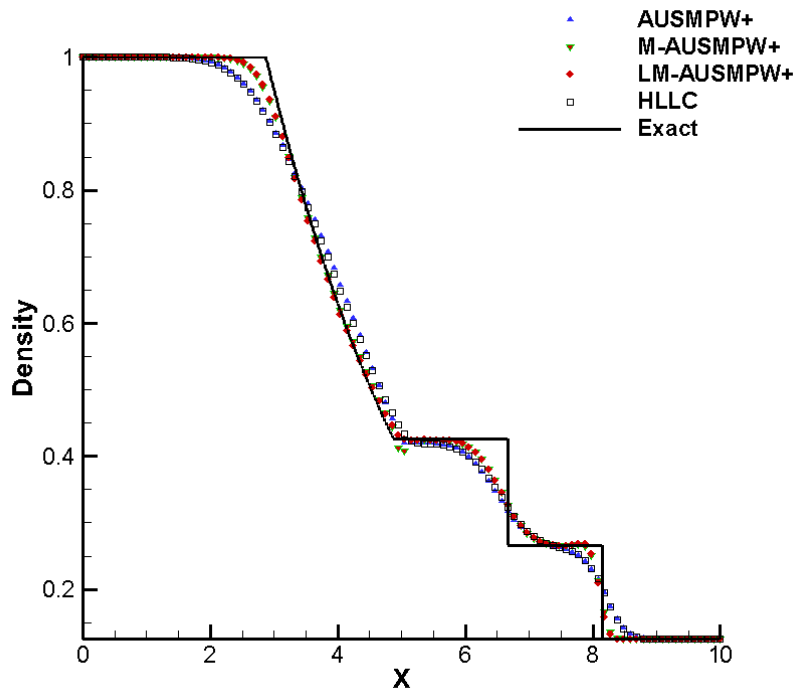
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4. **Conclusion**

Conclusion

◆ Summary

- ■ **LM-AUSMPW+ : extension of M-AUSMPW+ scheme for all-speed flow**
  - ◆ Accuracy problem at the transonic region is cured
  - ◆ Pressure splitting function is newly defined to be continuous and differentiable

  - ◆ LM-AUSMPW+ scheme is extended to the unstructured grid
  - ◆ Superbee limiter value is replaced by MLP-u1 value

Thank you for listening

Receding shock

◆ **Supersonic inverse flow problem**

■ **Initial condition**

$$(\rho_L, u_L, p_L) = (1, -2, 0.4), \qquad (\rho_R, u_R, p_R) = (1, 2, 0.4)$$

■ *Asdf*

Receding shock

◆ **Subsonic inverse flow problem**

■ **Initial condition**

$$(\rho_L, u_L, p_L) = (1, -0.5, 0.4), \qquad (\rho_R, u_R, p_R) = (1, 0.5, 0.4)$$

■ *Asdf*

Riemann problem 5

◆ **Shock entropy wave interaction**

■ Initial condition

$$(\rho_L, u_L, p_L) = \left(\frac{27}{7}, \frac{4\sqrt{35}}{9}, \frac{31}{3} \right), \qquad (\rho_R, u_R, p_R) = (1 + \epsilon sin(kx), 0, 1)$$

■ TVD Runge − Kutta 3$^{\text{th}}$ order method with Δt = 0.005 until t = 1.8sec

■ [0,10] with 100 points, 1$^{\text{st}}$ order recons*truction* ***without re − evaluation process***

Riemann problem 5

◆ **Shock entropy wave interaction**

■ Initial condition

$$(\rho_L, u_L, p_L) = \left(\frac{27}{7}, \frac{4\sqrt{35}}{9}, \frac{31}{3}\right), \qquad (\rho_R, u_R, p_R) = (1 + \epsilon sin(kx), 0, 1)$$

■ TVD Runge − Kutta 3th order method with $\Delta t = 0.005$ until t = 1.8sec

■ [0,10] with 100 points, 1st order recons*truction **with re − evaluation process**

Riemann problem 6

◆ **Receding supersonic expansion**

■ Initial condition

$$(\rho_L, u_L, p_L) = (1, -2, 0.4), \qquad (\rho_R, u_R, p_R) = (1, 2, 0.4)$$

■ TVD Runge − Kutta 3th order method with $\Delta t = 0.005$ until t = 1.8sec

■ [0,10] with 100 points, 1st order recons$truction$ $\boldsymbol{with\ re-evaluation\ process}$

2D bump

◆ **Result**