

# **Improvement on M-AUSMPW+ scheme for all speed flow**

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# Table of Contents

**1.** Introduction

**2.** Modification for all-speed

**3.** Numerical result

**4.** Conclusion

# Table of Contents

## **1.** Introduction

# Introduction

## ◆ Characteristics which flow solver needs to have

Robustness & Accuracy

Multi-physics

Turbulent flow

Plasma

Ablation

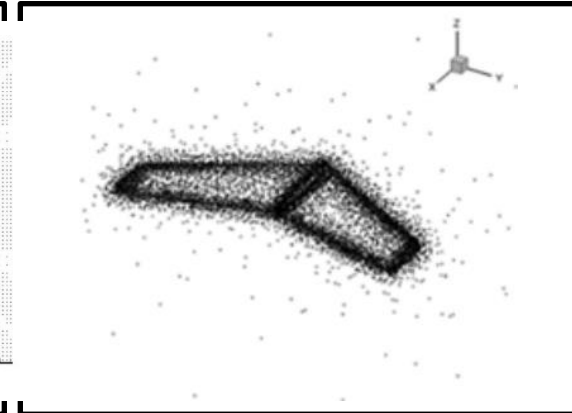
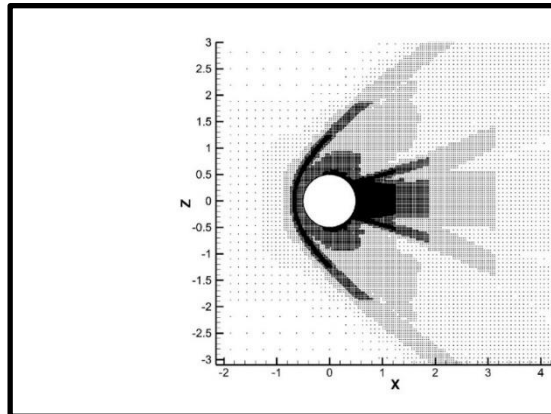
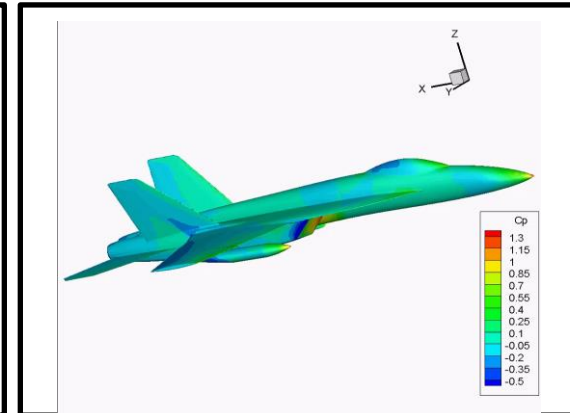
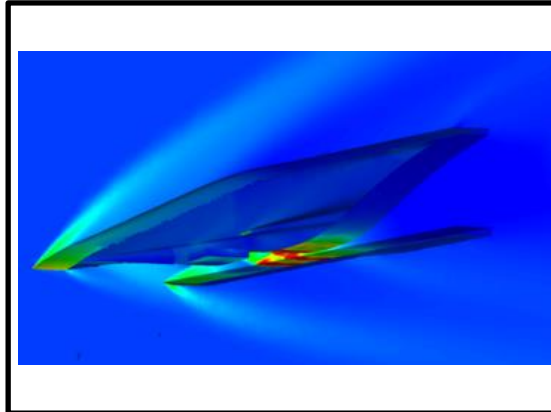
**All-speed flow**

Moving object analysis

Efficiency

Mesh optimization

Fast convergence

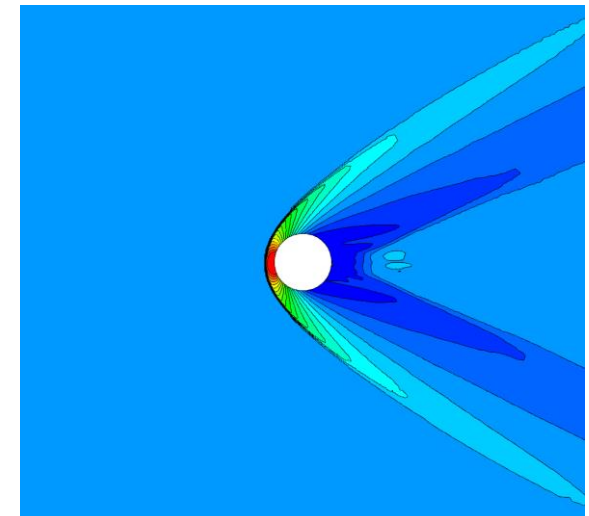
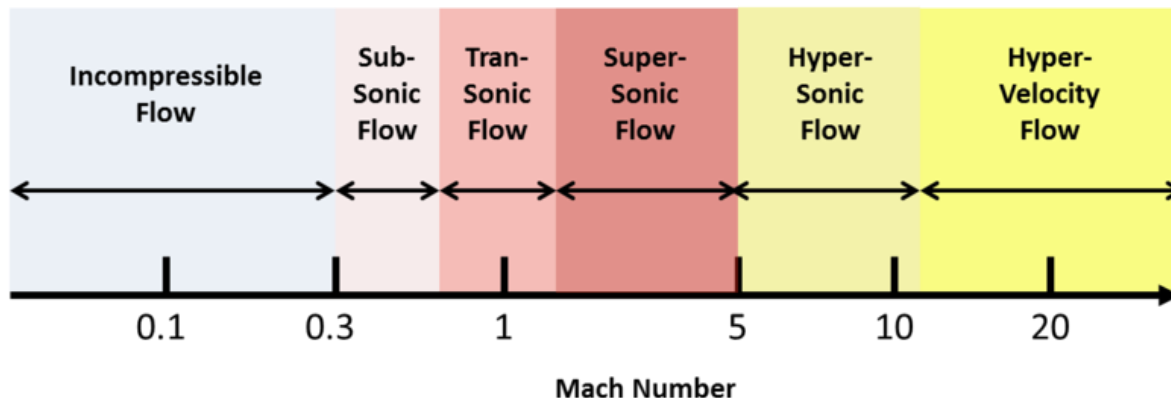


# Introduction

## ◆ All-speed flow

- The mathematical properties of the equations vary depending on the flow velocity range.
- It is common for users to use single method for the whole flow field.
- Most flux schemes which successfully solve the accuracy problem at supersonic speed **have difficulties in obtaining low Mach number flow solution.**

Mach Number Flow Regimes



# Table of Contents

## 2

### **Modification for all-speed**

- Governing equation
- Baseline scheme
- Asymptotic analysis
  - For governing equation
  - For discretized equation
- Improvement on M-AUSMPW+

# Governing equations

## ◆ Euler equations

- Conservative form is used for compressible flow

$$\frac{\partial q}{\partial t} + \nabla \cdot f_c(q) = 0$$

$$q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e_t \end{bmatrix}, \quad f_c(q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uH \end{bmatrix} \hat{i} + \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vH \end{bmatrix} \hat{j}$$

- Nondimensionalization

$$\rho^* = \frac{\rho}{\rho_\infty}, u^* = \frac{u}{u_\infty}, v^* = \frac{v}{v_\infty}, x^* = \frac{x}{l_\infty},$$
$$y^* = \frac{y}{l_\infty}, p^* = \frac{p}{\rho_\infty c_\infty^2}, e^* = \frac{e}{c_\infty^2}, t^* = \frac{t}{u_\infty / l_\infty}$$

$$\frac{\partial \rho^*}{\partial t^*} + \nabla(\rho^* u^*) = 0$$

$$\frac{\partial}{\partial t^*} \rho^* u^* + \nabla(\rho^* u^* u^*) = -\frac{1}{M_\infty^2} \nabla p^*$$

$$\frac{\partial}{\partial t^*} \rho^* e^* + \nabla(\rho^* e^* u^* + p^* u^*) = 0$$

# Baseline scheme: AUSMPW+/M-AUSMPW+

## ◆ AUSMPW+

- Control convection property by considering both left and right states across strong shock ► numerical oscillation/overshoot is cured

$$F_{\frac{1}{2}} = \bar{M}_L^+ c_{\frac{1}{2}} \Phi_L + \bar{M}_R^- c_{\frac{1}{2}} \Phi_R + P_L^+ p_L + P_R^- p_R$$

$$\text{if } M_L^+ + M_R^- \geq 0,$$

$$\bar{M}_L^+ = M_L^+ + M_R^- \cdot [(1-w)(1+f_R) - f_L]$$

$$\bar{M}_R^- = M_R^- \cdot w(1+f_R)$$

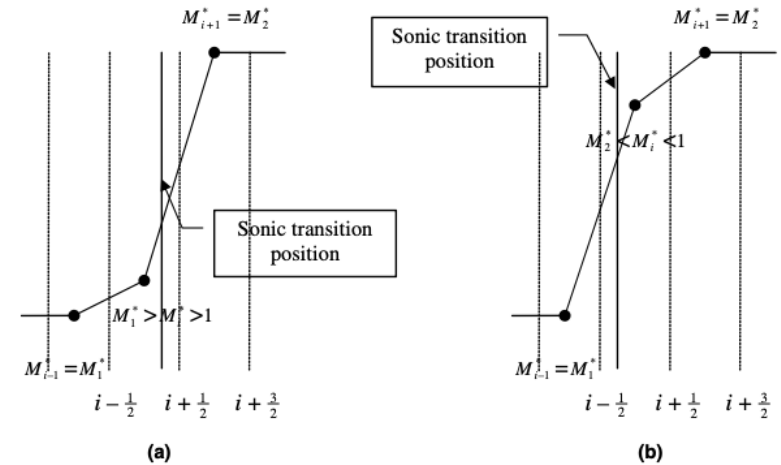
$$\text{if } M_L^+ + M_R^- < 0,$$

$$\bar{M}_L^+ = M_L^+ \cdot w(1+f_L)$$

$$\bar{M}_R^- = M_R^- + M_L^+ \cdot [(1-w)(1+f_L) - f_R]$$

$$f_{L,R} = \left( \frac{p_{L,R}}{p_s} - 1 \right) \min \left( 1, \frac{\min(p_{1,L}, p_{1,R}, p_{2,L}, p_{2,R})}{\min(p_L, p_R)} \right)$$

$$w = 1 - \min \left( \frac{p_R}{p_L}, \frac{p_L}{p_R} \right)^3$$





# Baseline scheme: AUSMPW+/M-AUSMPW+

## ◆ M-AUSMPW+

$$F_{\frac{1}{2}} = \bar{M}_L^+ c_{\frac{1}{2}} \Phi_{L,\frac{1}{2}} + \bar{M}_R^- c_{\frac{1}{2}} \Phi_{R,\frac{1}{2}} + P_L^+ p_{L,\frac{1}{2}} + P_R^- p_{R,\frac{1}{2}}$$

- ◆ R1) The region of continuity can be distinguished from discontinuity
- ◆ R2) Monotonic condition should be satisfied
- ◆ R3) Convective quantity should maintain upwind characteristic in supersonic flow

$$\Phi_{L,\frac{1}{2}} = \Phi_L + \frac{\max(0, (\Phi_R - \Phi_L)(\Phi_{L,sup} - \Phi_L))}{(\Phi_R - \Phi_L)|\Phi_{L,sup} - \Phi_L|} \min \left[ a \frac{|\Phi_R - \Phi_L|}{2}, |\Phi_{L,sup} - \Phi_L| \right]$$
$$\Phi_{R,\frac{1}{2}} = \Phi_R + \frac{\max(0, (\Phi_L - \Phi_R)(\Phi_{R,sup} - \Phi_R))}{(\Phi_L - \Phi_R)|\Phi_{R,sup} - \Phi_R|} \min \left[ a \frac{|\Phi_L - \Phi_R|}{2}, |\Phi_{R,sup} - \Phi_R| \right]$$
$$a = 1 - \min(1, \max(|M_L|, |M_R|))^2$$

- ◆ Modification of pressure splitting function which is aimed to improve accuracy in steady shock discontinuity

$$\text{If } M_i^* > 1, M_{i+1}^* < 1 \text{ and } 0 < M_i^* M_{i+1}^* < 1$$

$$P_{i+1}^- = \max(0, \min \left( 0.5, 1 - \frac{\rho_i U_i (U_i - U_{i+1}) + p_i}{p_{i+1}} \right))$$

$$\text{If } M_i^* > -1, M_{i+1}^* < -1 \text{ and } 0 < M_i^* M_{i+1}^* < 1$$

$$P_i^+ = \max(0, \min \left( 0.5, 1 - \frac{\rho_{i+1} U_{i+1} (U_{i+1} - U_i) + p_{i+1}}{p_i} \right))$$

# Asymptotic analysis for governing equations

## ◆ Asymptotic analysis for Euler equations

$$\rho^* = \rho_0^* + M_\infty \rho_1^* + M_\infty^2 \rho_2^* + \dots$$

$$p^* = p_0^* + M_\infty p_1^* + M_\infty^2 p_2^* + \dots$$

$$e^* = e_0^* + M_\infty e_1^* + M_\infty^2 e_2^* + \dots$$

$$u^* = u_0^* + M_\infty u_1^* + M_\infty^2 u_2^* + \dots$$

$$v^* = v_0^* + M_\infty v_1^* + M_\infty^2 v_2^* + \dots$$

1) Order of  $1/M_\infty^2$   
 $\nabla p_0^* = 0$

2) Order of  $1/M_\infty$   
 $\nabla p_1^* = 0$

3) Order of 1

$$\frac{\partial \rho_0^*}{\partial t^*} + \nabla(\rho_0^* u_0^*) = 0$$
$$\frac{\partial}{\partial t^*} \rho_0^* u_0^* + \nabla(\rho_0^* u_0^* u_0^*) = -\nabla p_2^*$$
$$\frac{\partial}{\partial t^*} \rho_0^* e_0^* + \nabla(\rho_0^* e_0^* u_0^* + p_0^* u_0^*) = 0$$

$$p^*(x, t) = p_0^*(t) + M_\infty^2 p_2^*(x, t) + \dots$$

# Behavior of M-AUSMPW+ on low speed

## ◆ Re-evaluation in continuous region

$$\Phi_L = \Phi_i + 0.5\phi(r_L)\Delta\Phi_{i-\frac{1}{2}}, \quad \Phi_R = \Phi_{i+1} - 0.5\phi(r_R)\Delta\Phi_{i+\frac{3}{2}}$$

$$r_L = \Delta\Phi_{i+1/2}/\Delta\Phi_{i-1/2}, \quad r_R = \Delta\Phi_{i+1/2}/\Delta\Phi_{i+3/2}$$

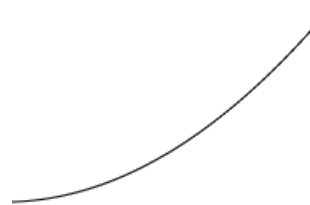
$$\Phi_{L,\frac{1}{2}} = \Phi_L + \frac{\max(0, (\Phi_R - \Phi_L)(\Phi_{L,sup} - \Phi_L))}{(\Phi_R - \Phi_L)|\Phi_{L,sup} - \Phi_L|} \min \left[ a \frac{|\Phi_R - \Phi_L|}{2}, |\Phi_{L,sup} - \Phi_L| \right]$$

$$\Phi_{R,\frac{1}{2}} = \Phi_R + \frac{\max(0, (\Phi_L - \Phi_R)(\Phi_{R,sup} - \Phi_R))}{(\Phi_L - \Phi_R)|\Phi_{R,sup} - \Phi_R|} \min \left[ a \frac{|\Phi_L - \Phi_R|}{2}, |\Phi_{R,sup} - \Phi_R| \right]$$

$$a = 1 - \min(1, \max(|M_L|, |M_R|))^2$$

- ◆ The value of  $\phi$  can be classified into three cases according to the profile of  $\phi$ . And behavior of re-evaluation is analyzed for each case.

- Concave/Convex condition
- Inflection point condition
- Local extrema



*Concave/convex*



*Inflection point*



*Local extrema*

# Behavior of M-AUSMPW+ on low speed

## ◆ Re-evaluation in continuous region

$$\Phi_L = \Phi_i + 0.5\phi(r_L)\Delta\Phi_{i-\frac{1}{2}}, \quad \Phi_R = \Phi_{i+1} - 0.5\phi(r_R)\Delta\Phi_{i+\frac{3}{2}}$$

$$r_L = \Delta\Phi_{i+1/2}/\Delta\Phi_{i-1/2}, \quad r_R = \Delta\Phi_{i+1/2}/\Delta\Phi_{i+3/2}$$

*Concave/Convex condition*

$$1) \frac{\partial\Phi}{\partial x} > 0, \frac{\partial^2\Phi}{\partial x^2} > 0$$

$$r_L > 1, 0 < r_R < \frac{1}{r_L}$$

$$2) \frac{\partial\Phi}{\partial x} > 0, \frac{\partial^2\Phi}{\partial x^2} < 0$$

$$0 < r_L < 1, 1 < r_R < \frac{1}{r_R}$$

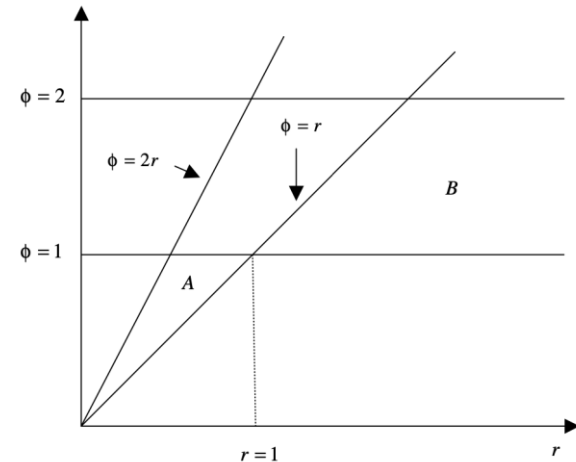
$$3) \frac{\partial\Phi}{\partial x} < 0, \frac{\partial^2\Phi}{\partial x^2} > 0$$

$$0 < r_L < 1, 1 < r_R < \frac{1}{r_R}$$

$$4) \frac{\partial\Phi}{\partial x} < 0, \frac{\partial^2\Phi}{\partial x^2} < 0$$

$$r_L > 1, 0 < r_R < \frac{1}{r_L}$$

$$\Phi_{L,R} = \Phi_{\frac{1}{2},real} + \Delta x^2 \phi'' \left[ \frac{1}{2} \phi'(1) - \frac{1}{3} \right] + O(\Delta x^3)$$



$$\Phi_{L,R,\frac{1}{2}} = \frac{\Phi_L + \Phi_R}{2} \rightarrow \text{Decreased Error!!}$$

# Behavior of M-AUSMPW+ on low speed

## ◆ Re-evaluation in continuous region

$$\begin{aligned}\Phi_L &= \Phi_i + 0.5\phi(r_L)\Delta\Phi_{i-\frac{1}{2}}, & \Phi_R &= \Phi_{i+1} - 0.5\phi(r_R)\Delta\Phi_{i+\frac{3}{2}} \\ r_L &= \Delta\Phi_{i+1/2}/\Delta\Phi_{i-1/2}, & r_R &= \Delta\Phi_{i+1/2}/\Delta\Phi_{i+3/2}\end{aligned}$$

*Inflection point*

$$1) r_L, r_R > 1$$

$$\phi'_L(1) = \phi'_R(1)$$

$$2) 0 < r_L, r_R < 1$$

$$\phi'_L(1) = \phi'_R(1)$$

$$\Phi_{L,R} = \Phi_{\frac{1}{2},real} + \Delta x^2 \phi'' \left[ \frac{1}{2} \phi'(1) - \frac{1}{3} \right] + O(\Delta x^3)$$

*Inflection point*

$$1) r_L, r_R > 1$$

$$\phi'_L(1) = \phi'_R(1)$$

$$2) 0 < r_L, r_R < 1$$

$$\phi'_L(1) = \phi'_R(1)$$

$$\Phi_{L,R,\frac{1}{2}} = \frac{\Phi_L + \Phi_R}{2} \rightarrow \text{The same Error!!}$$

# Behavior of M-AUSMPW+ on low speed

## ◆ Re-evaluation in continuous region

$$\Phi_L = \Phi_i + 0.5\phi(r_L)\Delta\Phi_{i-\frac{1}{2}}, \quad \Phi_R = \Phi_{i+1} - 0.5\phi(r_R)\Delta\Phi_{i+\frac{3}{2}}$$

$$r_L = \Delta\Phi_{i+1/2}/\Delta\Phi_{i-1/2}, \quad r_R = \Delta\Phi_{i+1/2}/\Delta\Phi_{i+3/2}$$

*Local extrema*

$$1) r_L < 0 < r_R$$

$$(\Phi_R - \Phi_L)(\Phi_{L,sup} - \Phi_L) = 0$$

$$(\Phi_L - \Phi_R)(\Phi_{R,sup} - \Phi_R) \geq 0$$

$$2) r_R < 0 < r_L$$

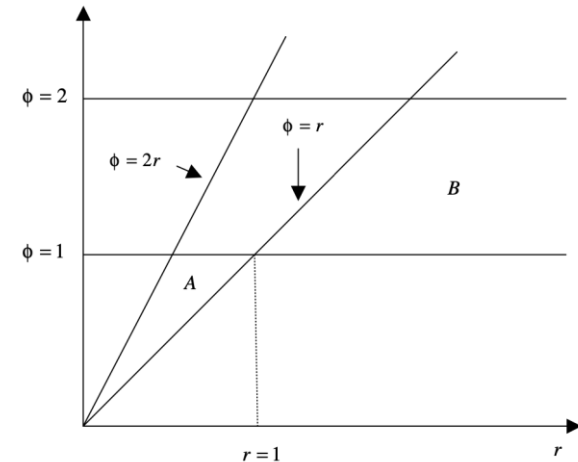
$$(\Phi_R - \Phi_L)(\Phi_{L,sup} - \Phi_L) \geq 0$$

$$(\Phi_L - \Phi_R)(\Phi_{R,sup} - \Phi_R) = 0$$

$$3) r_L = r_R = 0$$

$$\Phi_i = \Phi_L = \Phi_R = \Phi_{i+1}$$

$$\Phi_{L,R} = \Phi_{\frac{1}{2},real} + \Delta x^2 \phi'' \left[ \frac{1}{2} \phi'(1) - \frac{1}{3} \right] + O(\Delta x^3)$$



$$\Phi_{L,R,\frac{1}{2}} = \frac{\Phi_L + \Phi_R}{2} \rightarrow \text{The same or small Error!!}$$

# Behavior of M-AUSMPW+ on low speed

## ◆ Discretization of M-AUSMPW+ as $M \rightarrow 0$

- Re-evaluation for convective quantities is performed and re-evaluated value is closer to real value regardless of profile of  $\Phi$

$$\Phi_{L,\frac{1}{2}} = \Phi_{R,\frac{1}{2}} = \frac{\Phi_L + \Phi_R}{2}$$

$$\Phi = (\rho, u, p)^T$$

- Apply M-AUSMPW+ to gently varied region to carry out asymptotic analysis

$$M^\pm = \pm \frac{1}{4}(M \pm 1)^2 \approx \frac{1}{2}M \pm \frac{1}{4}$$

$$P^\pm = \frac{1}{4}(M \pm 1)^2(2 \mp M) \approx \pm \frac{3}{4}M + \frac{1}{2}$$

$$w = 1 - \min\left(\frac{p_R}{p_L}, \frac{p_L}{p_R}\right)^3 \approx 0$$

$$f_{L,R} = \left(\frac{p_{L,R}}{p_s} - 1\right) \min\left(1, \frac{\min(p_{1,L}, p_{1,R}, p_{2,L}, p_{2,R})}{\min(p_L, p_R)}\right) \approx \frac{p_{L,R}}{p_s} - 1$$

$$\hat{m}_{M-AUSMPW+} = M_L^+ c_1 \frac{\rho_L + \rho_R}{2} + M_R^- c_1 \frac{\rho_L + \rho_R}{2}$$

	(i, j+1)	
(i-1, j)	(i, j)	(i+1, j)
	(i, j-1)	

# Behavior of M-AUSMPW+ on low speed

## ◆ Discretization of M-AUSMPW+ as $M \rightarrow 0$

### ■ Continuity equation

$$\delta \frac{\partial}{\partial t} \rho_i + \frac{1}{2} \sum_l \left[ \frac{u_i \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \frac{u_l \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \left| \frac{u_i \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \frac{u_l \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} \right| \right] \\ + \frac{1}{2} \sum_l \left[ \frac{u_i \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \frac{u_l \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} - \left| \frac{u_i \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \frac{u_l \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} \right| \right] = 0$$

### ■ Momentum equation -x

$$\delta \frac{\partial}{\partial t} \rho_i u_i + \frac{1}{2} \sum_l \frac{u_i + u_l}{2} \left[ \frac{u_i \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \frac{u_l \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \left| \frac{u_i \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \frac{u_l \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} \right| \right] \\ + \frac{1}{2} \sum_l \frac{u_i + u_l}{2} \left[ \frac{u_i \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \frac{u_l \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} - \left| \frac{u_i \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \frac{u_l \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} \right| \right] \\ \frac{1}{2M_\infty^2} \sum_l p_l (\vec{n}_{il})_x + \frac{3}{4M_\infty} \sum_l \left[ \frac{u_i \cdot \vec{n}_{x,il}}{c_{ij}} \frac{p_i + p_l}{2} - \frac{u_l \cdot \vec{n}_{x,il}}{c_{ij}} \frac{p_i + p_l}{2} \right] = 0$$



# Behavior of M-AUSMPW+ on low speed

## ◆ Discretization of M-AUSMPW+ as $M \rightarrow 0$

### ■ Momentum equation -y

$$\begin{aligned} & \delta \frac{\partial}{\partial t} \rho_i v_i + \frac{1}{2} \sum_l \frac{v_i + v_l}{2} \left[ \frac{u_i \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \frac{u_l \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \left| \frac{u_i \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \frac{u_l \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} \right| \right] \\ & + \frac{1}{2} \sum_l \frac{v_i + v_l}{2} \left[ \frac{u_i \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \frac{u_l \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} - \left| \frac{u_i \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \frac{u_l \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} \right| \right] \\ & \frac{1}{2M_\infty^2} \sum_l p_l (\vec{n}_{il})_y + \frac{3}{4M_\infty} \sum_l \left[ \frac{u_i \cdot \vec{n}_{y,il}}{c_{ij}} \frac{p_i + p_l}{2} - \frac{u_l \cdot \vec{n}_{y,il}}{c_{ij}} \frac{p_i + p_l}{2} \right] = 0 \end{aligned}$$

### ■ Energy equation

$$\begin{aligned} & \delta \frac{\partial}{\partial t} \rho_i e_{t,i} + \frac{1}{2} \sum_l \frac{e_{t,i} + e_{t,l}}{2} \left[ \frac{u_i \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \frac{u_l \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \left| \frac{u_i \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \frac{u_l \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} \right| \right] \\ & + \frac{1}{2} \sum_l \frac{e_{t,i} + e_{t,l}}{2} \left[ \frac{u_i \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \frac{u_l \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} - \left| \frac{u_i \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \frac{u_l \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} \right| \right] \\ & + \sum_l \frac{p_i + p_l}{2} \left[ \frac{u_i \cdot \vec{n}_{il}}{2} + \frac{u_l \cdot \vec{n}_{il}}{2} \right] = 0 \end{aligned}$$

# Behavior of M-AUSMPW+ on low speed

## ◆ Discretization of M-AUSMPW+ as $M \rightarrow 0$

### ■ AUSMPW+ .vs. M-AUSMPW+

$$\hat{m}_{AUSMPW+} = M_L^+ c_{\frac{1}{2}} \rho_L + M_R^- c_{\frac{1}{2}} \rho_R$$

$$\hat{m}_{M-AUSMPW+} = M_L^+ c_{\frac{1}{2}} \frac{\rho_L + \rho_R}{2} + M_R^- c_{\frac{1}{2}} \frac{\rho_L + \rho_R}{2}$$

AUSMPW+:

$$\delta \frac{\partial}{\partial t} \rho_i + \sum_l \left[ \frac{u_i \cdot \vec{n}_{il}}{2} \rho_i + \frac{u_l \cdot \vec{n}_{il}}{2} \rho_l + \frac{1}{4M_\infty} (\rho_i - \rho_l) \right] = 0$$

$$\delta \frac{\partial}{\partial t} \rho_i u_i + \sum_l \frac{1}{\rho_\infty u_\infty^2} (\rho u u)_{il} + \frac{1}{2M_\infty^2} \sum_l p_l (\vec{n}_{il})_x + \frac{3}{4M_\infty} \sum_l \left[ \frac{u_i \cdot \vec{n}_{x,il}}{c_{ij}} p_i - \frac{u_l \cdot \vec{n}_{x,il}}{c_{ij}} p_l \right] = 0$$

$M - AUSMPW+$ :

$$\delta \frac{\partial}{\partial t} \rho_i + \sum_l \left[ \frac{u_i \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \frac{u_l \cdot \vec{n}_{il}}{2} \frac{\rho_i + \rho_l}{2} + \frac{1}{4M_\infty} \left( \frac{\rho_i + \rho_l}{2} - \frac{\rho_i + \rho_l}{2} \right) \right] = 0$$

$$\delta \frac{\partial}{\partial t} \rho_i u_i + \sum_l \frac{1}{\rho_\infty u_\infty^2} (\rho u u)_{il} + \frac{1}{2M_\infty^2} \sum_l p_l (\vec{n}_{il})_x + \frac{3}{4M_\infty} \sum_l \left[ \frac{u_i \cdot \vec{n}_{x,il}}{c_{ij}} \frac{p_i + p_l}{2} - \frac{u_l \cdot \vec{n}_{x,il}}{c_{ij}} \frac{p_i + p_l}{2} \right] = 0$$

# Behavior of M-AUSMPW+ on low speed

## ◆ Discretization of M-AUSMPW+ as $M \rightarrow 0$

### ■ AUSMPW+ .vs. M-AUSMPW+

#### ◆ Asymptotic analysis for discretized equations

$$\rho^* = \rho_0^* + M_\infty \rho_1^* + M_\infty^2 \rho_2^* + \dots$$

$$p^* = p_0^* + M_\infty p_1^* + M_\infty^2 p_2^* + \dots$$

⋮

*AUSMPW+*

1) Order of  $1/M_\infty^2$

$$p_{0,i+1,j}^* - p_{0,i-1,j}^* = 0$$

$$p_{0,i,j+1}^* - p_{0,i,j-1}^* = 0$$

2) Order of  $1/M_\infty$

$$\frac{1}{2} \sum_l p_{1,l} (\overrightarrow{n_{il}})_x - \frac{3}{4} \sum_l \left[ \frac{u_{0,l} \cdot \overrightarrow{n_{x,il}}}{c_{ij}} p_{0,l} \right] = 0$$

$$\frac{1}{2} \sum_l p_{1,l} (\overrightarrow{n_{il}})_y - \frac{3}{4} \sum_l \left[ \frac{u_{0,l} \cdot \overrightarrow{n_{y,il}}}{c_{ij}} p_{0,l} \right] = 0$$

*M-AUSMPW+*

1) Order of  $1/M_\infty^2$

$$p_{0,i+1,j}^* - p_{0,i-1,j}^* = 0$$

$$p_{0,i,j+1}^* - p_{0,i,j-1}^* = 0$$

2) Order of  $1/M_\infty$

$$p_{1,i+1,j}^* - p_{1,i-1,j}^* = 0$$

$$p_{1,i,j+1}^* - p_{1,i,j-1}^* = 0$$

$$p^*(x, t) = p_0^*(t) + M_\infty^2 p_2^*(x, t) + \dots - \text{satisfied?}$$

# Behavior of M-AUSMPW+ on low speed

## ◆ Characteristics of M-AUSMPW+ as $M \rightarrow 0$

- “ $p^*(x, t) = p_0^*(t) + M_\infty^2 p_2^*(x, t) + \dots$ ” is *satisfied for any  $\alpha$  and  $\beta$*

$$M^\pm = \begin{cases} \pm \frac{1}{4}(M \pm 1)^2 \pm \beta(M^2 - 1)^2 & |M| \leq 1 \\ \frac{1}{2}(M \pm |M|) & |M| > 1 \end{cases}$$

$$P^\pm = \begin{cases} \frac{1}{4}(M \pm 1)^2(2 \mp M) \pm \alpha M(M^2 - 1)^2 & |M| \leq 1 \\ \frac{1}{2}(1 \pm \text{sign}(|M|)) & |M| > 1 \end{cases}$$

- ◆ R1) Splitting function should satisfy  $f = f^+ + f^-$
- ◆ R2)  $f^+$  and  $f^-$  should mimic the symmetry with respect to  $M$ .

$$M^\pm = \pm \frac{1}{4}(M \pm 1)^2 \pm \beta(M^2 - 1)^2 = \pm a_0 + \frac{1}{2}M + \dots$$

$$P^\pm = \frac{1}{4}(M \pm 1)^2(2 \mp M) \pm \alpha M(M^2 - 1)^2 = \frac{1}{2} \pm b_1 M + \dots$$

$$\hat{m}_{M-AUSMPW+} = (M_L^+ + M_R^-) c_1 \frac{\rho_L + \rho_R}{2}$$

$$p_{\frac{1}{2}, M-AUSMPW+} = (P_L^+ + P_R^-) \frac{p_L + p_R}{2}$$

# Improvement on M-AUSMPW+

## ◆ Instability due to zero numerical dissipation

- Discretized equations imply that M-AUSMPW+ produces zero numerical dissipation at low speed ( $M \rightarrow 0$ )

$$p_{\frac{1}{2}} = \frac{p_L + p_R}{2} + D$$

$$D_{AUSMPW+} \neq 0$$

$$D_{M-AUSMPW+} = 0$$

- Numerical dissipation is added to pressure term to cure the instability due to zero numerical dissipation

- ◆ C1) Asymptotic characteristics of governing equations should be satisfied in additional numerical dissipation.
- ◆ C2) Supersonic characteristics of baseline scheme should be maintained.
- ◆ C3) The size of numerical dissipation is determined according to Mach number region.

# Improvement on M-AUSMPW+

## ◆ Instability due to zero numerical dissipation

- Discretized equations imply that M-AUSMPW+ produces zero numerical dissipation at low speed

$$p_{\frac{1}{2}} = \frac{p_L + p_R}{2} + \left( P_L^+ p_L + P_R^- p_R - \frac{p_L + p_R}{2} \right) \min(1, M) \\ + P_L^+ P_R^- M^2 \left( 1 - \min(1, \max(|M_L|, |M_R|)) \right)^2 (p_R - p_L)$$

- ◆  $P_L^+ P_R^-$  is designed to switch off numerical dissipation at supersonic flow
- ◆  $M^2$  is for satisfying the asymptotic behavior of governing equations
- ◆  $\min(1, M)$  scales down numerical dissipation at low speed region

# Table of Contents

## 3. Numerical result

# Riemann problem 1

## ◆ Stationary contact discontinuity

- Initial condition

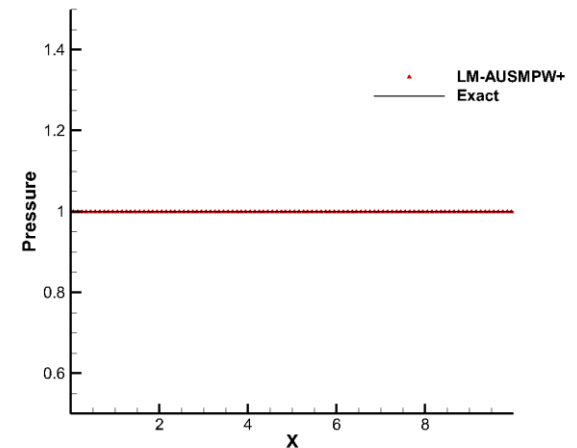
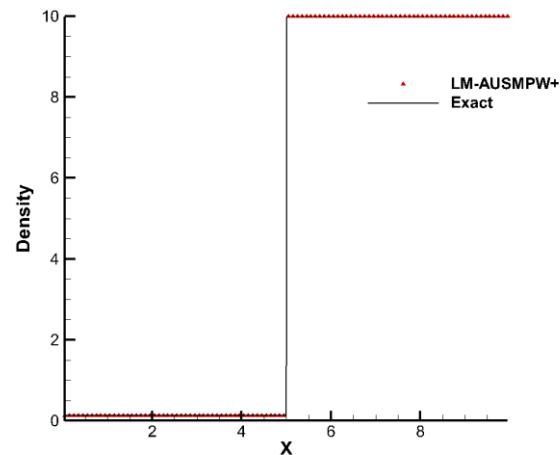
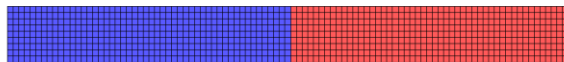
$$(\rho_L, u_L, p_L) = (0.125, 0, 1), \quad (\rho_R, u_R, p_R) = (10, 0, 1)$$

- Discontinuity in density can be preserved by zero mass/pressure flux.
- *Discontinuity line is preserved although numerical dissipation is added.*

$$\bar{M}_L^+ = M_L^+ + M_R^- \cdot [(1 - w)(1 + f_R) - f_L] = 0$$

$$\bar{M}_R^- = M_R^- \cdot w(1 + f_R) = 0$$

$$p_{\frac{1}{2}} = p_L = p_R$$





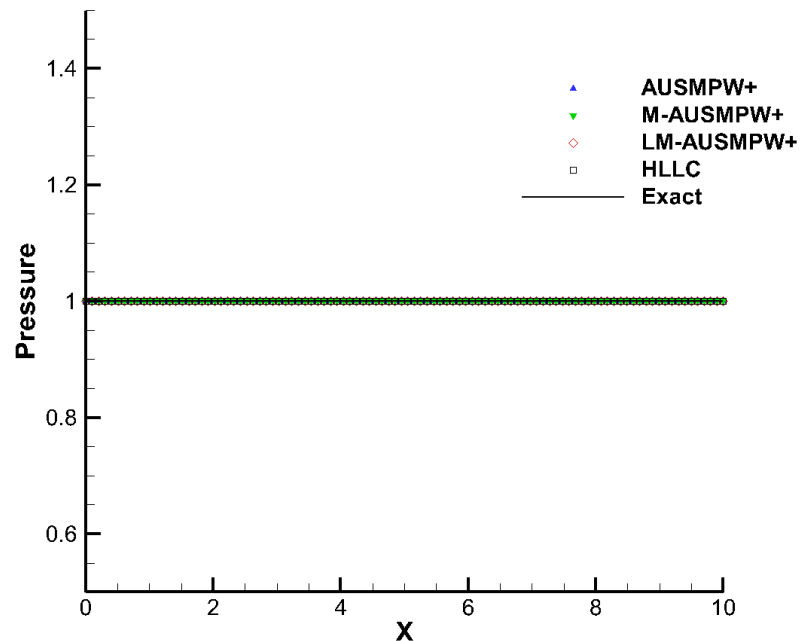
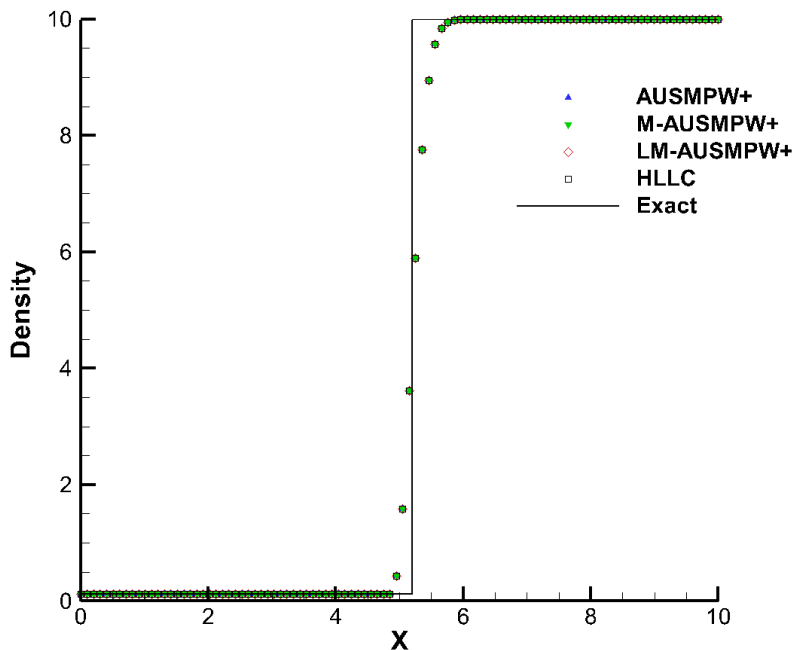
# Riemann problem 2

## ◆ Moving contact discontinuity

- Initial condition

$$(\rho_L, u_L, p_L) = (0.125, 0.1125, 1), \quad (\rho_R, u_R, p_R) = (10, 0.1125, 1)$$

- TVD Runge – Kutta 3<sup>th</sup> order method with  $\Delta t = 0.005$  until  $t = 2.5\text{sec}$
- $[0,10]$  with 100 points, 1<sup>st</sup> order reconstruction *without re – evaluation process*



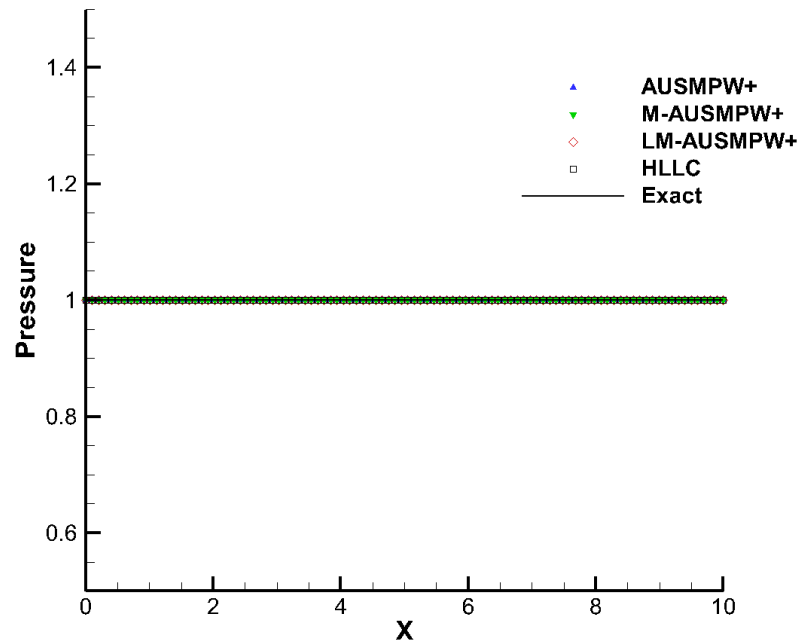
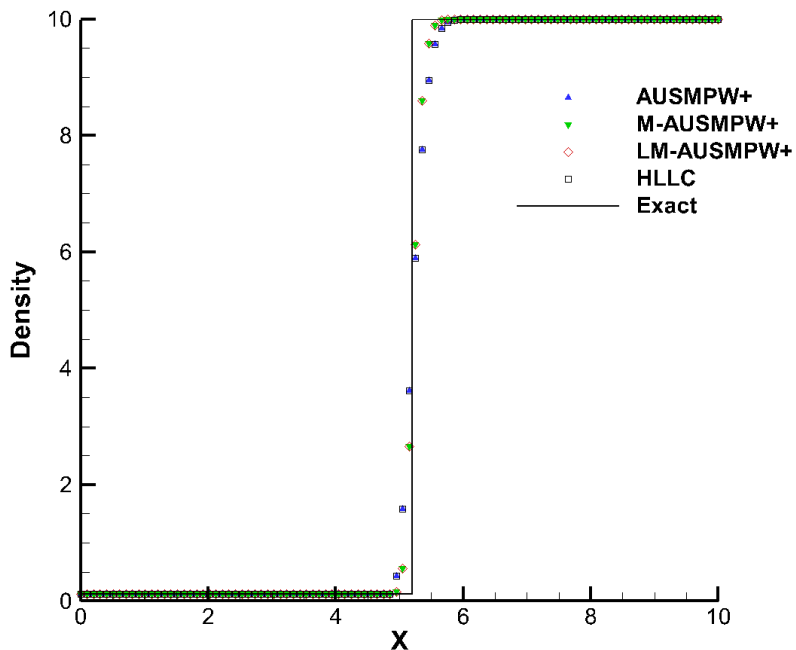
# Riemann problem 2

## ◆ Moving contact discontinuity

- Initial condition

$$(\rho_L, u_L, p_L) = (0.125, 0.1125, 1), \quad (\rho_R, u_R, p_R) = (10, 0.1125, 1)$$

- TVD Runge – Kutta 3<sup>th</sup> order method with  $\Delta t = 0.005$  until  $t = 2.5\text{sec}$
- $[0,10]$  with 100 points, 1<sup>st</sup> order reconstruction *with re – evaluation process*



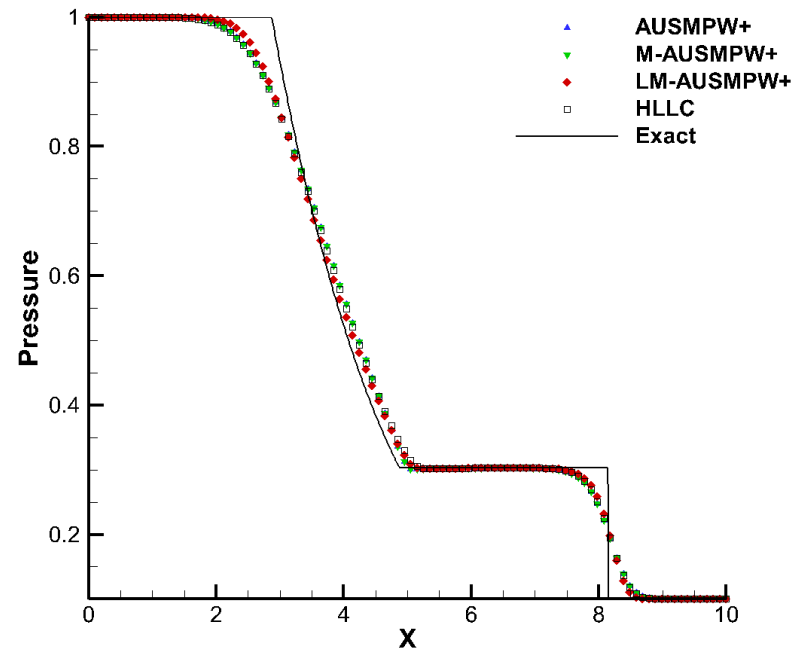
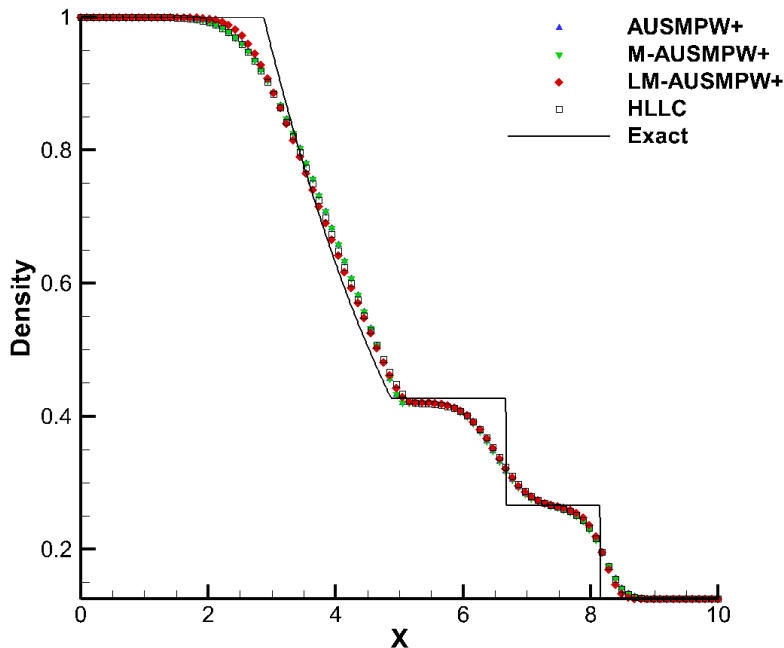
# Riemann problem 3

## ◆ Sod problem

- Initial condition

$$(\rho_L, u_L, p_L) = (1, 0, 1), \quad (\rho_R, u_R, p_R) = (0.125, 0, 0.1)$$

- TVD Runge – Kutta 3<sup>th</sup> order method with  $\Delta t = 0.005$  until  $t = 1.8\text{sec}$
- $[0,10]$  with 100 points, 1<sup>st</sup> order reconstruction *without re – evaluation process*



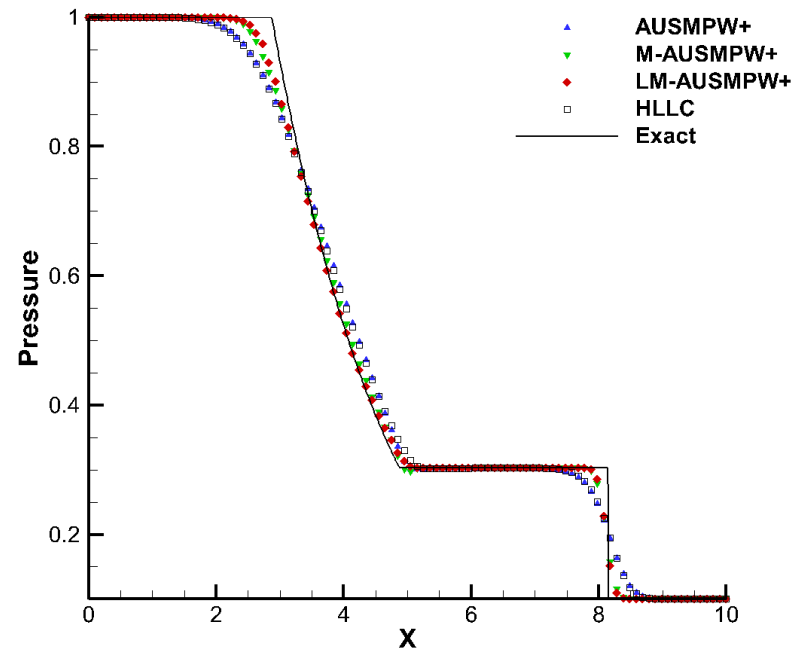
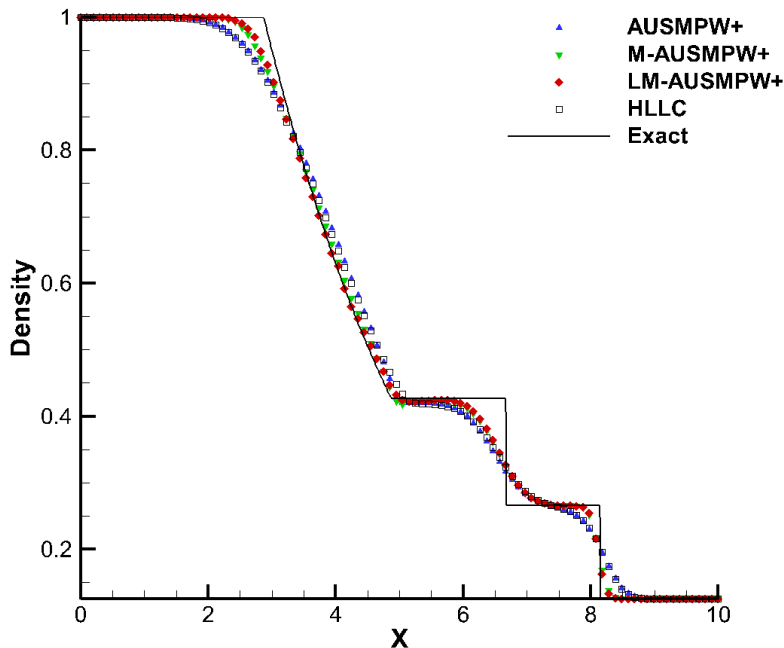
# Riemann problem 3

## ◆ Sod problem

- Initial condition

$$(\rho_L, u_L, p_L) = (1, 0, 1), \quad (\rho_R, u_R, p_R) = (0.125, 0, 0.1)$$

- TVD Runge – Kutta 3<sup>th</sup> order method with  $\Delta t = 0.005$  until  $t = 1.8\text{sec}$
- $[0,10]$  with 100 points, 1<sup>st</sup> order reconstruction *with re – evaluation process*



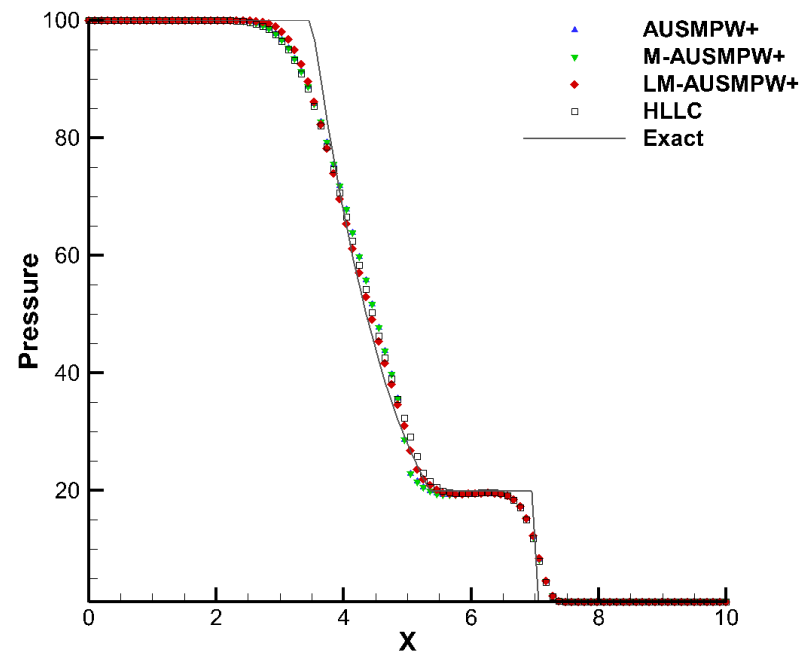
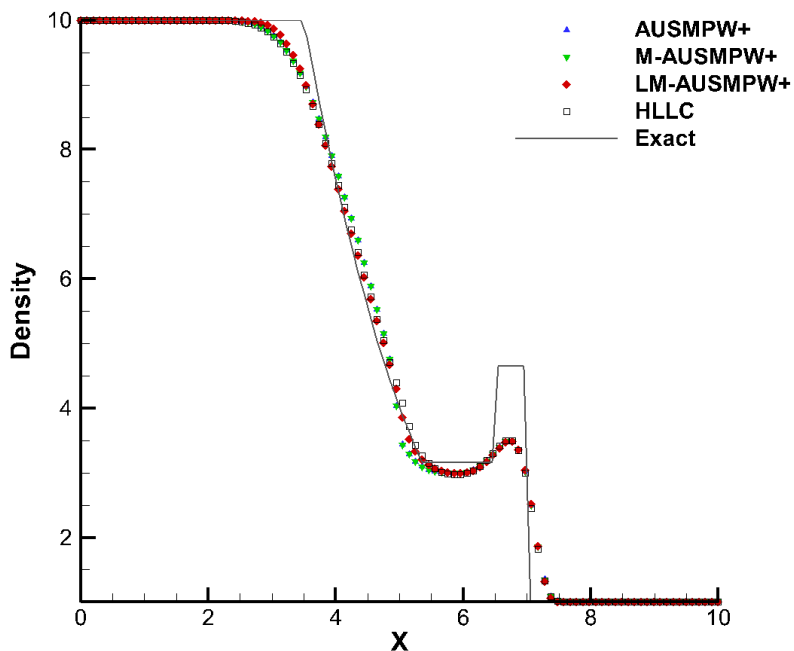
# Riemann problem 4

## Strong shock problem

- Initial condition

$$(\rho_L, u_L, p_L) = (10, 0, 100), \quad (\rho_R, u_R, p_R) = (1, 0, 1)$$

- TVD Runge – Kutta 3<sup>th</sup> order method with  $\Delta t = 0.005$  until  $t = 0.4\text{sec}$
- $[0,10]$  with 100 points, 1<sup>st</sup> order reconstruction *without re – evaluation process*



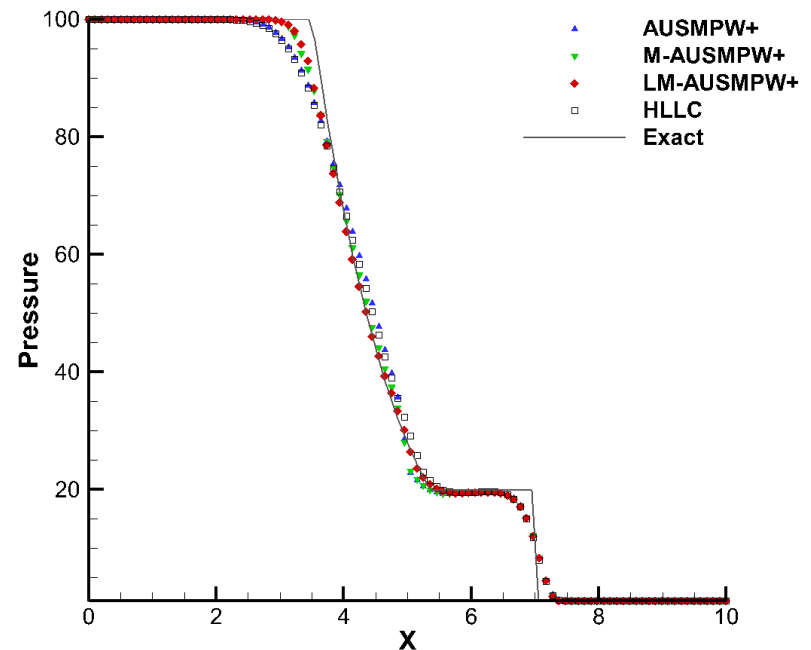
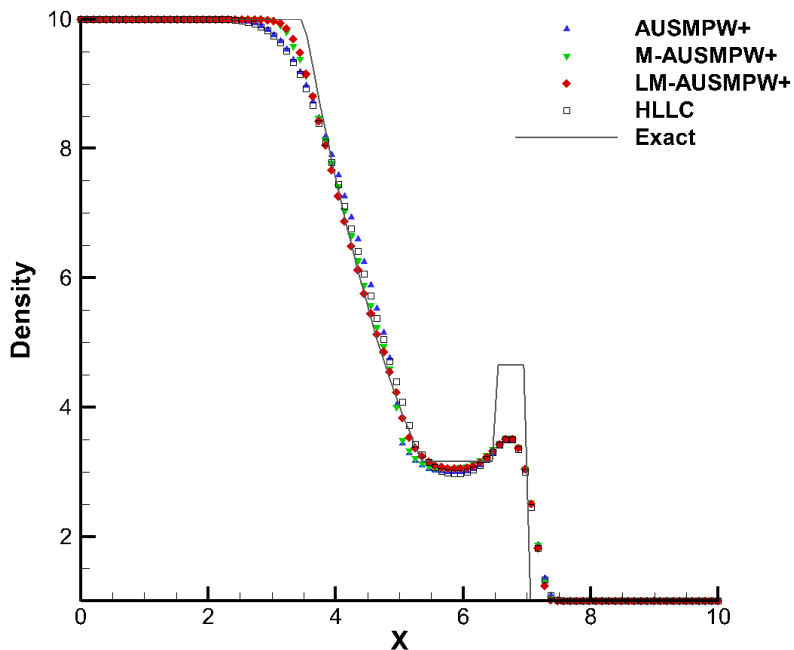
# Riemann problem 4

## Strong shock problem

- Initial condition

$$(\rho_L, u_L, p_L) = (10, 0, 100), \quad (\rho_R, u_R, p_R) = (1, 0, 1)$$

- TVD Runge – Kutta 3<sup>th</sup> order method with  $\Delta t = 0.005$  until  $t = 0.4\text{sec}$
- $[0,10]$  with 100 points, 1<sup>st</sup> order reconstruction *with re – evaluation process*



# Vortex flow

## ◆ Isentropic vortex flow

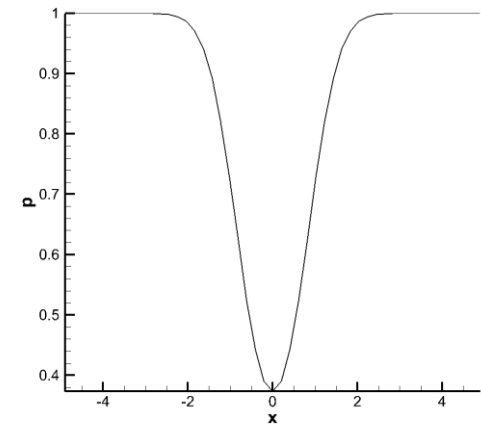
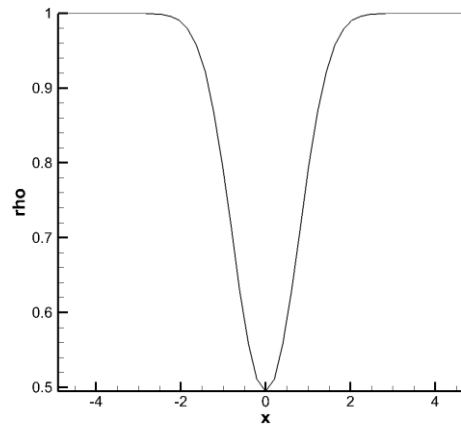
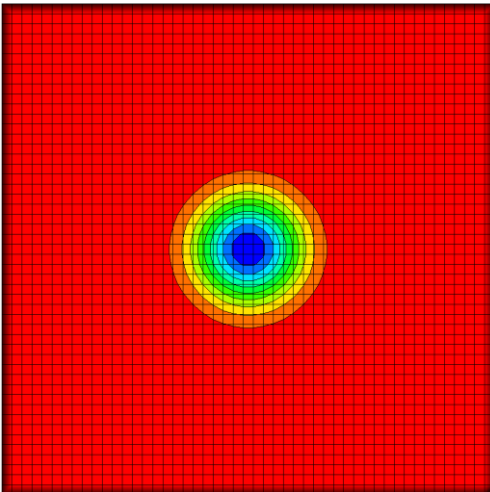
### ■ *Initial condition*

$$u = -\frac{5}{2\pi}(y - y_0)e^{\frac{1-r^2}{2}}$$

$$v = +\frac{5}{2\pi}(x - x_0)e^{\frac{1-r^2}{2}}$$

$$\delta T = -\frac{5^2(\gamma - 1)}{8\gamma\pi^2}e^{1-r^2}$$

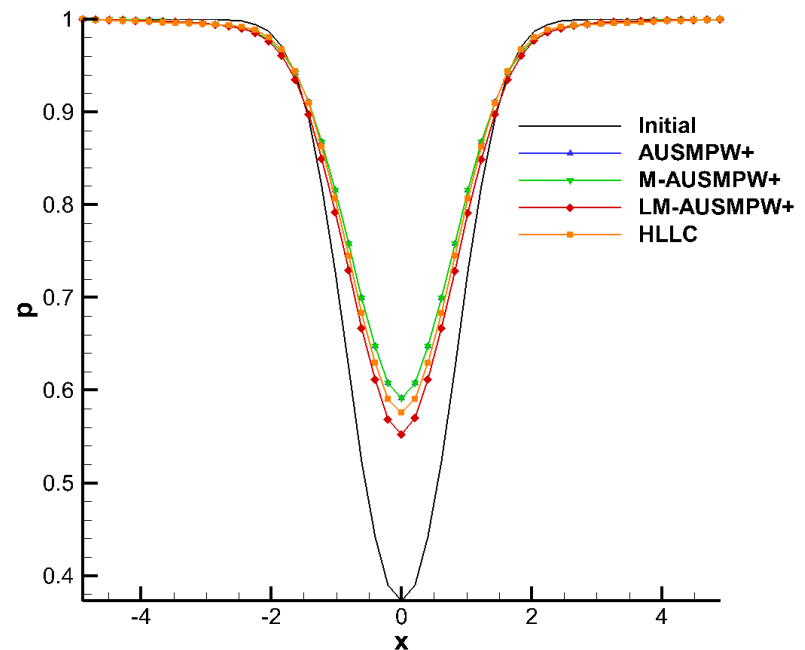
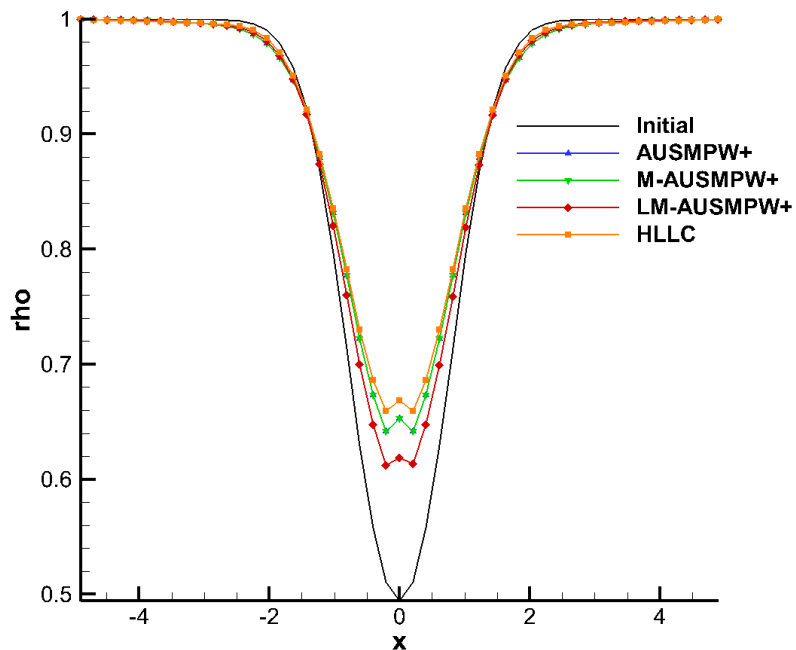
### ■ *Initial profile should be conserved since isentropic condition is assumed.*



# Vortex flow

## ◆ Isentropic vortex flow

- TVD Runge – Kutta 3<sup>th</sup> order method with  $\Delta t = 0.01$  until  $t = 2\text{sec}$
- $50 \times 50$  points, 1<sup>st</sup> order reconstruction **without re – evaluation process**

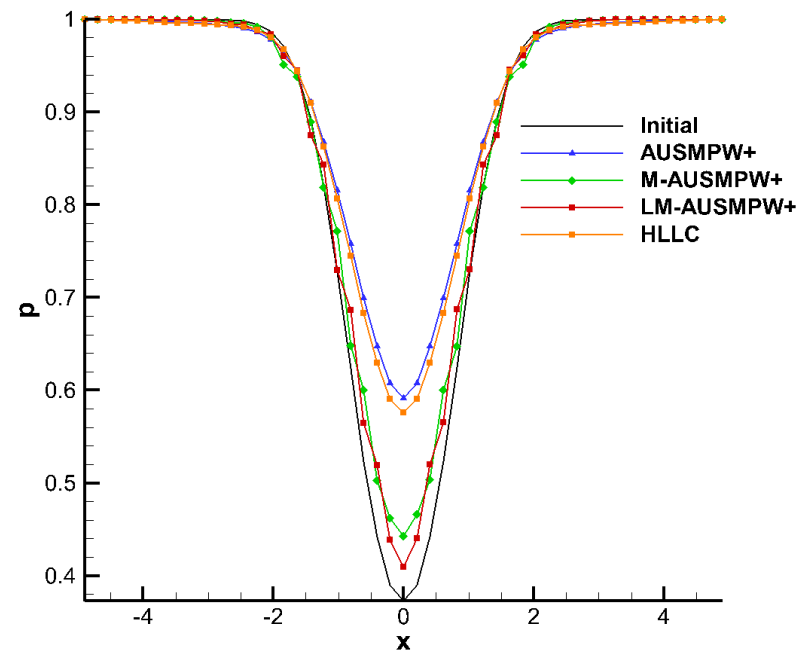
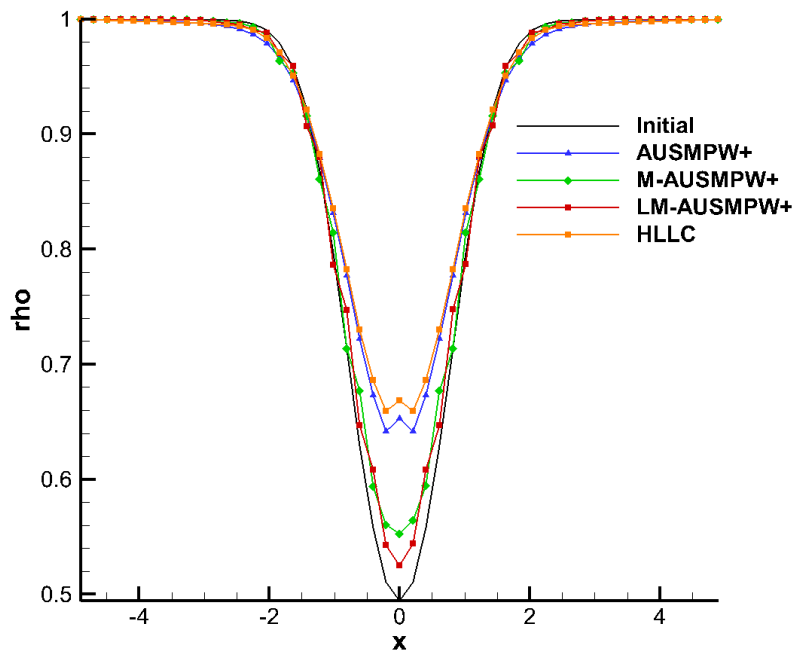




# Vortex flow

## ◆ Isentropic vortex flow

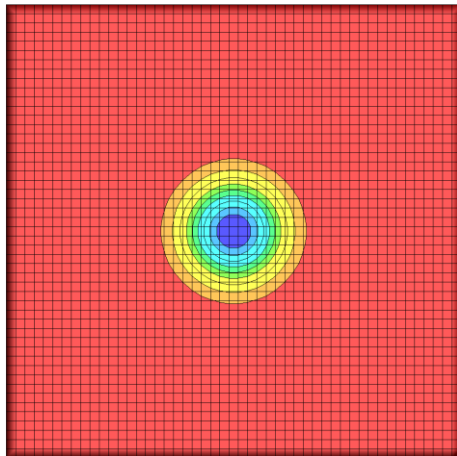
- TVD Runge – Kutta 3<sup>th</sup> order method with  $\Delta t = 0.01$  until  $t = 2\text{sec}$
- $50 \times 50$  points, 1<sup>st</sup> order reconstruction **with re – evaluation process**



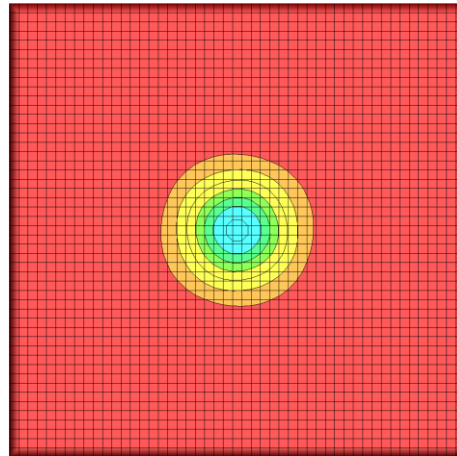
# Vortex flow

## ◆ Isentropic vortex flow

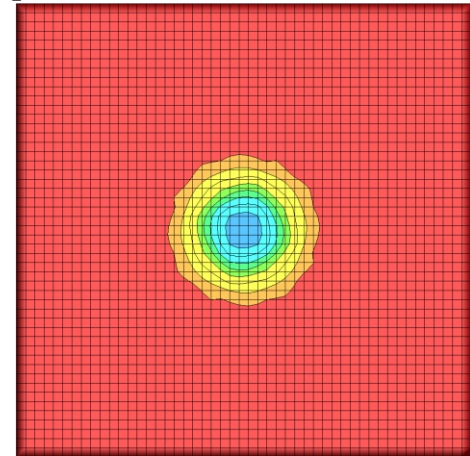
- Comparison of density contour *with re – evaluation process*



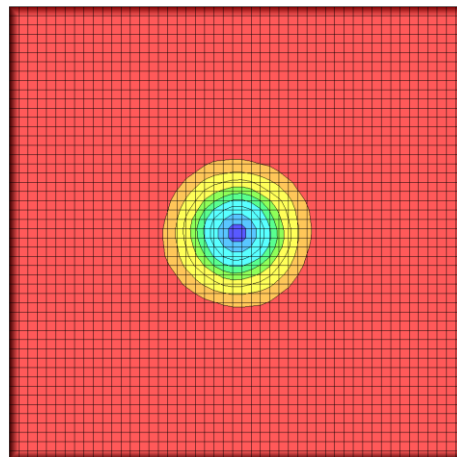
*Initial*



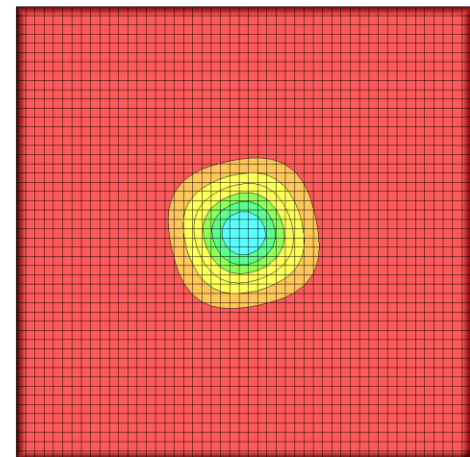
*AUSMPW +*



*M - AUSMPW +*



*LM - AUSMPW +*



*HLLC*

# Table of Contents

## 4. Conclusion

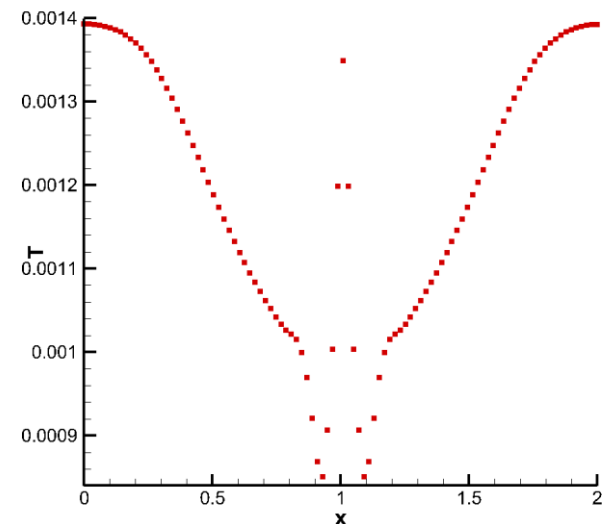
# Conclusion

## ◆ Summary

- LM-AUSMPW+ : extension of M-AUSMPW+ scheme for all-speed flow
  - ◆ Asymptotic analysis for the governing equations
  - ◆ Apply discretization to the equations by assuming 2D cartesian grid and first order FVM context
  - ◆ Asymptotic analysis to the discretized equations focusing on the difference between AUSMPW+ and M-AUSMPW+
  - ◆ Numerical dissipation of M-AUSMPW+ is controlled so that asymptotic behavior can be satisfied.

## ◆ Future work

- More numerical tests needed
  - ◆ Ex) 2D bump...
- Unresolved problem: “overheating problem”
  - ◆ Entropy problem in energy equation



The background features a faint, light blue line graph with several data points connected by lines, set against a white background. On the left side, there is a vertical blue gradient bar. At the bottom left, there is a dark grey rectangle with a yellow triangle pointing right, and below it, a series of horizontal blue lines.

**Thank you for listening**

# Receding shock

## ◆ Supersonic inverse flow problem

### ■ Initial condition

$$(\rho_L, u_L, p_L) = (1, -2, 0.4), \quad (\rho_R, u_R, p_R) = (1, 2, 0.4)$$

### ■ *Asdf*

# Receding shock

## ◆ Subsonic inverse flow problem

### ■ Initial condition

$$(\rho_L, u_L, p_L) = (1, -0.5, 0.4), \quad (\rho_R, u_R, p_R) = (1, 0.5, 0.4)$$

### ■ *Asdf*

# Riemann problem 5

## ◆ Shock entropy wave interaction

- Initial condition

$$(\rho_L, u_L, p_L) = \left( \frac{27}{7}, \frac{4\sqrt{35}}{9}, \frac{31}{3} \right), \quad (\rho_R, u_R, p_R) = (1 + \epsilon \sin(kx), 0, 1)$$

- TVD Runge – Kutta 3<sup>th</sup> order method with  $\Delta t = 0.005$  until  $t = 1.8\text{sec}$
- $[0,10]$  with 100 points, 1<sup>st</sup> order reconstruction ***without re – evaluation process***



# Riemann problem 5

## ◆ Shock entropy wave interaction

- Initial condition

$$(\rho_L, u_L, p_L) = \left( \frac{27}{7}, \frac{4\sqrt{35}}{9}, \frac{31}{3} \right), \quad (\rho_R, u_R, p_R) = (1 + \epsilon \sin(kx), 0, 1)$$

- TVD Runge – Kutta 3<sup>th</sup> order method with  $\Delta t = 0.005$  until  $t = 1.8\text{sec}$
- $[0,10]$  with 100 points, 1<sup>st</sup> order reconstruction ***with re – evaluation process***

# Riemann problem 6

## ◆ Receding supersonic expansion

- Initial condition

$$(\rho_L, u_L, p_L) = (1, -2, 0.4), \quad (\rho_R, u_R, p_R) = (1, 2, 0.4)$$

- TVD Runge – Kutta 3<sup>th</sup> order method with  $\Delta t = 0.005$  until  $t = 1.8\text{sec}$
- $[0,10]$  with 100 points, 1<sup>st</sup> order reconstruction ***with re – evaluation process***

# 2D bump

◆ Result